

# A Global Stability Result in Utility-Based Congestion Control

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**Abstract**—In this paper, we introduce a new utility-based congestion control algorithm, M-REM (modified-REM), and show its convergence. The algorithm is a slight modification of the so-called REM (random exponential marking) gradient projection algorithm, in which all buffers in the network are cleared when equilibrium source rates are reached. By defining the “price” of a particular link as a weighted sum of the Lagrange multiplier and buffer length associated with that link, we show that M-REM is globally asymptotically stable, while preserving the decentralized nature of REM.

## I. INTRODUCTION

Primary objective of network congestion control is to regulate the transmission rate of each user in the network in order to obtain and maintain a satisfactory quality of service (QoS) for all users. In utility-based congestion control, this is achieved through a utility maximization problem, in which each user has a utility function reflecting its valuation of transmitting at a certain rate. The aggregate user utility is then maximized over transmission rates of all users subject to the link-capacity constraints. The significance of the utility maximization model is that TCP Reno [1] and Vegas [2], as well as several other Internet congestion control algorithms can be interpreted within this model by choosing appropriate utility functions. This approach has been extensively studied in several papers, such as [3], [4], [5] [6], [7], [8]. Solving the utility maximization problem directly requires coordination among possibly all sources, and hence is impractical for the networks. However, there exist decentralized solutions which only use partial information from the network. One way of obtaining a decentralized solution is to look at the dual of the maximization problem. In [6], it has been shown that this leads to a decentralized algorithm that consists of a link algorithm that updates a congestion measure, also called “price”, based on the excess capacity at that link, and a source algorithm that adapts the source rate to congestion in its path. This algorithm, which is shown to converge, has a drawback in that the backlog can be quite large in the equilibrium. To remedy this problem, in [9], REM (Random Exponential Marking) algorithm has been introduced, which ensures that the buffer is cleared when the equilibrium is reached. The convergence of this more appealing algorithm for a multi-link network has not been established yet. In [10], the continuous time version of REM has been shown to be globally stable using a Lyapunov argument, but for the

original discrete-time case, a proof of stability is available only for the special case of a single-link, see [11], [12]. In this paper, we propose a congestion control algorithm similar to REM in that when the source rates reach the equilibrium, all buffers in the network are cleared. By incorporating buffer length variation into the price vector, we show that the modified version of the REM algorithm, M-REM, is globally asymptotically stable. Meanwhile, the decentralized nature of REM is preserved in M-REM.

The rest of the paper is organized as follows. In Section II, we describe the utility-based network resource-allocation problem, and its decentralized solution REM. The modified REM algorithm, M-REM, is introduced in Section 3, and its stability analysis is carried out in Section 4. The paper ends with the concluding remarks of Section 5, in which we also discuss some future research directions.

## II. UTILITY-BASED CONGESTION CONTROL

### A. Network model

Consider a network  $\mathcal{N}$  that consists of a set  $\mathcal{L} = \{1, \dots, L\}$  of links of capacity  $c_l$ ,  $l \in \mathcal{L}$ . The network is shared by a set  $\mathcal{S} = \{1, \dots, S\}$  of sources. Source  $s$  transmits at rate  $x_s$  using a set  $\mathcal{L}_s \subseteq \mathcal{L}$  of links. The routing matrix  $R$ , of dimension  $L \times S$ , is defined by  $R_{ls} = \mathcal{X}_{s \in \mathcal{S}_l}$ , where  $\mathcal{S}_l$  is the set of sources using link  $l$ , and  $\mathcal{X}_A$  denotes the indicator function of the set  $A$ .

The rate  $x_s$  satisfies  $m_s \leq x_s \leq M_s$ , where  $m_s \geq 0$  and  $M_s < \infty$  are the minimum and maximum transmission rates, respectively. When transmitting at rate  $x_s$ , source  $s$  attains a utility of  $U_s(x_s)$ . It is assumed that the utility functions  $U_s$  are strictly concave increasing and twice continuously differentiable. Associated with each link  $l$  there is a buffer with occupancy  $b_l$ .

We assume an underlying discrete time structure with no delay and synchronous updates. Let  $t$  denote the discrete time unit, and assume that fluid approximation for queue lengths holds. Then, the buffer occupancy  $b_l^{(t)}$  at link  $l$  at time  $t$  evolves according to

$$b_l^{(t+1)} = \left[ b_l^{(t)} + \sum_{s \in \mathcal{S}_l} x_s^{(t)} - c_l \right]^+ \quad (1)$$

where the notation  $[x]^+$  is the projection of  $x$  onto the positive orthant.

### B. The optimization problem

Our objective (the primal problem) is to choose the source rates  $x_s$  so as to solve the following optimization problem:

$$\max_{m_s \leq x_s \leq M_s} \sum_{s \in \mathcal{S}} U_s(x_s) \quad (2)$$

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subject to capacity constraints:

$$\sum_{s \in \mathcal{S}_l} x_s \leq c_l \quad (3)$$

This flow control problem was first posed in [3], and solved in [4] using techniques of constrained optimization. The problem can also be solved using a penalty function approach, see [7]. Note that a unique maximizer exists, called the primal optimal solution, since the objective function is strictly concave, and the feasible solution set is compact. Even though the objective function is separable in  $x_s$ , the source rates are coupled by the constraint (3). Thus, solving the primal problem directly requires coordination among possibly all sources and is impractical for networks. Looking at the dual of the optimization problem (2)-(3), one can obtain a gradient projection algorithm in the dual variables that lends itself into a decentralized implementation [6]. The Lagrangian dual of this problem is

$$\min_{p_l \geq 0} \max_{m_s \leq x_s \leq M_s} \sum_{s \in \mathcal{S}} U_s(x_s) - \sum_{l \in \mathcal{L}} p_l \left( \sum_{s \in \mathcal{S}_l} x_s - c_l \right) \quad (4)$$

where associated with each link there is a dual variable  $p_l$ , termed as ‘‘price’’ of link  $l$ . The solution of this dual optimization problem can be obtained using a gradient projection algorithm. The basic algorithm given in [6] works as follows. At times  $t = 1, 2, \dots$ , link  $l$  receives rates from all sources  $s \in \mathcal{S}_l$  that go through it, and it calculates the aggregate link rate

$$y_l^{(t)} := \sum_{s \in \mathcal{S}_l} x_s^{(t)}$$

With this information, it computes the so-called link price  $p_l^{(t)}$ , which is updated according to

$$p_l^{(t+1)} = \left[ p_l^{(t)} + \gamma \left( \sum_{s \in \mathcal{S}_l} x_s^{(t)} - c_l \right) \right]^+ \quad (5)$$

where  $\gamma > 0$  is a step size. Note that this update can be interpreted as a gradient step in the minimization of (4). This price is communicated to all sources  $s \in \mathcal{S}_l$  that use link  $l$ . Source  $s$ , upon receiving the aggregate price

$$P_s^{(t)} := \sum_{l \in \mathcal{L}_s} p_l^{(t)}$$

of all links in its path, chooses a new transmission rate  $x_s^{(t)}$  by solving the maximization problem

$$\max_{m_s \leq x_s^{(t)} \leq M_s} U_s(x_s^{(t)}) - x_s^{(t)} P_s^{(t)} \quad (6)$$

whose solution by the Kuhn-Tucker theorem is given by

$$x_s^{(t)} = [U_s'^{-1}(P_s^{(t)})]_{m_s}^{M_s} \quad (7)$$

where  $[x]_a^b = \min\{\max\{x, a\}, b\}$ . Here  $U_s'^{-1}$  denote the inverse functions of the marginal utility; they exist and are strictly decreasing because  $U_s$  are strictly concave increasing.

In [6], it has been shown that this distributed algorithm converges to the solution of the primal optimization problem

(2)-(3) if the curvatures of  $U_s$  are bounded away from zero, i.e.  $-U_s''(x_s) \geq 1/\rho_s > 0$ ,  $m_s \leq x_s \leq M_s$ , and if the step size  $\gamma$  satisfies  $0 < \gamma < 2/\rho \bar{S} \bar{L}$ , where  $\rho = \max_{s \in \mathcal{S}} \rho_s$ ,  $\bar{L} = \max_{s \in \mathcal{S}} |\mathcal{L}_s|$ , and  $\bar{S} = \max_{l \in \mathcal{L}} |\mathcal{S}_l|$ .

### C. REM

Note that, in the gradient-projection algorithm (5), prices integrate excess capacity, which is exactly what the backlogs do:

$$b_l^{(t+1)} = \left[ b_l^{(t)} + \sum_{s \in \mathcal{S}_l} x_s^{(t)} - c_l \right]^+$$

Comparing this equation with (5), we see that backlogs are related to prices by  $b_l(t) = \gamma^{-1} p_l(t)$ . Thus, the backlog can be quite large in equilibrium since  $\gamma > 0$  is typically small. To remedy this problem, in [9] the price adjustment (5) is modified to

$$p_l^{(t+1)} = \left[ p_l^{(t)} + \gamma(\alpha b_l^{(t)} + \sum_{s \in \mathcal{S}_l} x_s^{(t)} - c_l) \right]^+ \quad (8)$$

where  $\alpha > 0$  is a small constant. This extra backlog term will, if equilibrium is achieved, guarantee that  $b_l = 0$ , i.e. the buffer is cleared. The update algorithm (8) is commonly referred to as REM, whose convergence has not been established yet. In [10], the continuous time version of REM has been shown to be globally stable, but for the original discrete-time case, a proof of stability is available only for the single-link case; see [11], [12].

## III. M-REM

### A. The algorithm

In this paper, we propose a new algorithm, M-REM, which is a slightly modified version of (7)-(8), which however will enable us to prove global asymptotic stability. First, we let  $q_l^{(t)} = \alpha b_l^{(t)}$ , and note that in terms of  $q_l^{(t)}$ , evolution (1) of the buffer occupancy can be written as

$$q_l^{(t+1)} = \left[ q_l^{(t)} + \alpha \left( \sum_{s \in \mathcal{S}_l} x_s^{(t)} - c_l \right) \right]^+$$

Next, we redefine the link price  $\pi_l^{(t)}$  as

$$\pi_l^{(t)} := p_l^{(t)} + q_l^{(t)} = p_l^{(t)} + \alpha b_l^{(t)},$$

and we require that this price, instead of  $p_l^{(t)}$ , be communicated to all sources  $s \in \mathcal{S}_l$  that use link  $l$ . Source  $s$ , upon receiving the aggregate price

$$\Pi_s^{(t)} := \sum_{l \in \mathcal{L}_s} \pi_l^{(t)}$$

of all links in its path, chooses a new transmission rate  $x_s^{(t)}$  by solving the modified maximization problem

$$\max_{m_s \leq x_s^{(t)} \leq M_s} U_s(x_s^{(t)}) - x_s^{(t)} \Pi_s^{(t)}$$

whose solution is given by

$$x_s^{(t)} = [U_s'^{-1}(\Pi_s^{(t)})]_{m_s}^{M_s} \quad (9)$$

The overall system is then described by the following set of equations

$$p_l^{(t+1)} = \left[ p_l^{(t)} + \gamma(q_l^{(t)} + y_l^{(t)} - c_l) \right]^+ \quad (10)$$

$$q_l^{(t+1)} = \left[ q_l^{(t)} + \alpha(y_l^{(t)} - c_l) \right]^+ \quad (11)$$

Let  $p = (p_l, l \in \mathcal{L}), q = (q_l, l \in \mathcal{L}), z = (p, q)$ , and  $x = (x_s, s \in \mathcal{S})$ . Let  $(p^*, q^*)$  be an equilibrium point of the system (10)-(11)<sup>1</sup>, and let  $\Pi^* = R^T \pi^* = R^T p^* = P^*$  be the equilibrium source prices,  $x^* = x(P^*)$  the equilibrium source rates, and  $y^* = R x^*$  the equilibrium link rates. Clearly, we must have that  $b^* = 0$ .  $p^* = \pi^*$ , on the other hand, need not be zero, indeed its nonzero components correspond to links where  $y_l^* = 0$ ; i.e. where the capacity constraint is active (bottleneck links). It has been shown in [6] that the equilibrium is a saddle point of (4); therefore it follows from duality theory that  $x^*$  must be the unique global optimum of the primal problem (2)-(3); therefore  $y^*, P^*$  are also unique. However,  $p^*$  need not be unique, because in general capacity constraints might not be independent. In order to obtain a unique equilibrium price, we make the assumption that the routing matrix  $R$  is of full row rank. This assumption guarantees that, for a given vector  $P$ , there is a unique vector  $p$  satisfying  $P = R^T p$ .

### B. Implementation

Source  $s$  needs to have access to the aggregate price  $\Pi_s^{(t)}$  of all links in its path to calculate its transmission rate according to (9). One way of communicating this information to source  $s$  is through marking. Suppose on link  $l$  we mark packets with probability  $1 - e^{-\pi_l^{(t)}}$ , hence the *random exponential marking* (REM). It is conceivable that source  $s$  measures the fraction  $f_s^{(t)}$  of unmarked packets in time slot  $t$ , which is given by

$$f_s^{(t)} = e^{-\sum_{l \in \mathcal{L}_s} \pi_l^{(t)}} = e^{-\Pi_s^{(t)}}$$

Thus, source  $s$  can estimate  $\Pi_s^{(t)}$  as

$$\Pi_s^{(t)} = -\ln f_s^{(t)}$$

The rate  $x_s^{(t)}$  of source  $s$  can then be calculated according to (9).

## IV. STABILITY ANALYSIS OF M-REM

We start our analysis by rewriting the M-REM update scheme in a more convenient form. First, let

$$\phi_l(z) = q_l + \sum_{s \in \mathcal{S}_l} x_s \left( \sum_{l \in \mathcal{L}_s} p_l + q_l \right) - c_l$$

$$\psi_l(z) = \sum_{s \in \mathcal{S}_l} x_s \left( \sum_{l \in \mathcal{L}_s} p_l + q_l \right) - c_l$$

Then, the update scheme (10)-(11) can be written as

$$p_l^{(t+1)} = \left[ p_l^{(t)} + \gamma \phi_l(z^{(t)}) \right]^+ =: T_{p_l}(z^{(t)})$$

$$q_l^{(t+1)} = \left[ q_l^{(t)} + \alpha \psi_l(z^{(t)}) \right]^+ =: T_{q_l}(z^{(t)})$$

<sup>1</sup>Note that equilibrium points of REM, and M-REM are the same.

or equivalently as a fixed-point map

$$z^{(t+1)} = T(z^{(t)})$$

where  $T_p = (T_{p_l}, l \in \mathcal{L}), T_q = (T_{q_l}, l \in \mathcal{L})$ , and  $T = (T_p, T_q)$ .

Let  $z \in \mathcal{R}_+^{2L}$ , and define for each link  $l$  the functions  $g_l(\tau) : [0, 1] \rightarrow \mathcal{R}_+, h_l(\tau) : [0, 1] \rightarrow \mathcal{R}_+$  by [14]

$$g_l(\tau) = [\tau p_l + (1 - \tau) p_l^* + \gamma \phi_l(\tau z + (1 - \tau) z^*)]^+$$

$$h_l(\tau) = [\tau q_l + (1 - \tau) q_l^* + \alpha \psi_l(\tau z + (1 - \tau) z^*)]^+$$

Let  $g = (g_l, l \in \mathcal{L}), h = (h_l, l \in \mathcal{L})$ . Note that, for a given  $z \in \mathcal{R}_+^{2L}$ , both  $g_l(\tau)$  and  $h_l(\tau)$  are differentiable except possibly at a single point.<sup>2</sup> We have

$$g_l(0) = T_{p_l}(z^*), \quad h_l(0) = T_{q_l}(z^*)$$

$$g_l(1) = T_{p_l}(z), \quad h_l(1) = T_{q_l}(z)$$

Now, observe that

$$\begin{aligned} & |T_{p_l}(z) - T_{p_l}(z^*)| = |g_l(1) - g_l(0)| \\ &= \left| \int_0^1 \frac{dg_l(\tau)}{d\tau} d\tau \right| \leq \int_0^1 \left| \frac{dg_l(\tau)}{d\tau} \right| d\tau \\ &\leq \max_{\tau \in [0,1]} \left| \frac{dg_l(\tau)}{d\tau} \right| \end{aligned}$$

and similarly for  $T_{q_l}$ :

$$|T_{q_l}(z) - T_{q_l}(z^*)| \leq \max_{\tau \in [0,1]} \left| \frac{dh_l(\tau)}{d\tau} \right|$$

Assuming that  $\gamma < 1/\bar{\mu} |\mathcal{S}_l| \bar{L}$  and  $\alpha < 1/\bar{\mu} |\mathcal{S}_l| \bar{L}$ , we can bound  $\left| \frac{dg_l(\tau)}{d\tau} \right|$ , and  $\left| \frac{dh_l(\tau)}{d\tau} \right|$  from above as follows

$$\begin{aligned} \left| \frac{dg_l(\tau)}{d\tau} \right| &\leq (1 - \gamma \mu |\mathcal{S}_l|) |p_l - p_l^*| \\ &\quad + \gamma \rho |\mathcal{S}_l| \bar{L} \sum_{l' \neq l \in \mathcal{L}} |p_{l'} - p_{l'}^*| \\ &\quad + \gamma (1 + \rho |\mathcal{S}_l| \bar{L}) \sum_{l \in \mathcal{L}} |q_l - q_l^*| \end{aligned}$$

$$\begin{aligned} \left| \frac{dh_l(\tau)}{d\tau} \right| &\leq (1 - \alpha \mu |\mathcal{S}_l|) |q_l - q_l^*| \\ &\quad + \alpha \rho |\mathcal{S}_l| \bar{L} \sum_{l' \neq l \in \mathcal{L}} |q_{l'} - q_{l'}^*| \\ &\quad + \alpha \rho |\mathcal{S}_l| \bar{L} \sum_{l \in \mathcal{L}} |p_l - p_l^*| \end{aligned}$$

Here,  $|\mathcal{S}_l|, |\mathcal{L}_s|$  denote the number of elements in the sets  $\mathcal{S}_l, \mathcal{L}_s$ , respectively,  $\rho_s, \mu_s$  are defined by

$$\frac{1}{\mu_s} \geq -U_s''(x_s) \geq \frac{1}{\rho_s} > 0, \quad m_s \leq x_s \leq M_s \quad (12)$$

$$\rho = \max_{s \in \mathcal{S}} \rho_s, \quad \mu = \min_{s \in \mathcal{S}} \mu_s, \quad \bar{\mu} = \max_{s \in \mathcal{S}} \mu_s$$

and  $\bar{L} = \max_{s \in \mathcal{S}} |\mathcal{L}_s|, \bar{S} = \max_{l \in \mathcal{L}} |\mathcal{S}_l|$ .

In other words, in addition to the lower bound on the curvatures of the utility functions  $U_s$ , we require that the

<sup>2</sup>To be more precise and rigorous, derivatives should be replaced by convex subgradients; but for simplicity, we ignore these issues in this version of the paper.

curvatures are upper bounded by a constant  $1/\mu_s$ . This guarantees that the distance between the price and buffer length updates and their equilibrium values at link  $l$  are contracting in the direction of  $|p_l - p_l^*|, |q_l - q_l^*|$ , respectively.

Let

$$\|z\| := \max_l \{|p_l|, |q_l|\}$$

denote the maximum norm, and let  $\gamma$  and  $\alpha$  satisfy:

$$0 < \gamma < \frac{1}{\bar{\mu}\bar{S}\bar{L}}, \quad 0 < \alpha < \frac{1}{\bar{\mu}\bar{S}\bar{L}} \quad (13)$$

We then have

$$\left| \frac{dg_l(\tau)}{d\tau} \right| \leq \max\{(1 - \gamma\mu), \gamma(1 + \rho\bar{S}\bar{L})\} \|z - z^*\|$$

and

$$\left| \frac{dh_l(\tau)}{d\tau} \right| \leq \max\{(1 - \alpha\mu), \alpha\rho\bar{S}\bar{L}\} \|z - z^*\|$$

Or, equivalently,

$$\begin{aligned} |T_{p_l}(z) - T_{p_l}(z^*)| &\leq \max\{(1 - \gamma\mu), \gamma(1 + \rho\bar{S}\bar{L})\} \|z - z^*\| \\ |T_{q_l}(z) - T_{q_l}(z^*)| &\leq \max\{(1 - \alpha\mu), \alpha\rho\bar{S}\bar{L}\} \|z - z^*\| \end{aligned}$$

Hence,

$$\begin{aligned} \|T(z) - T(z^*)\| &\leq \max\{(1 - \gamma\mu), (1 - \alpha\mu), \gamma(1 + \rho\bar{S}\bar{L}), \alpha\rho\bar{S}\bar{L}\} \|z - z^*\| \\ &=: r \|z - z^*\| \end{aligned}$$

Finally, we pick  $r < 1$ , i.e.

$$\max\{(1 - \gamma\mu), (1 - \alpha\mu), \gamma(1 + \rho\bar{S}\bar{L}), \alpha\rho\bar{S}\bar{L}\} < 1$$

to achieve contraction, so that

$$\begin{aligned} \|T^{(t)}(z) - T(z^*)\| &\leq r^{(t)} \|z - z^*\| \rightarrow 0 \\ \Rightarrow T^{(t)}(z) &\rightarrow T(z^*) = z^* \end{aligned}$$

In terms of the algorithm parameters, to guarantee the condition  $r < 1$ , it is sufficient to choose

$$0 < \gamma < \min\left\{\frac{1}{\bar{\mu}}, \frac{1}{1 + \rho\bar{S}\bar{L}}\right\}$$

$$0 < \alpha < \min\left\{\frac{1}{\bar{\mu}}, \frac{1}{\rho\bar{S}\bar{L}}\right\}$$

However, we also need to satisfy (13). Therefore, a sufficient condition for contraction becomes

$$0 < \gamma < \min\left\{\frac{1}{\bar{\mu}\bar{S}\bar{L}}, \frac{1}{1 + \rho\bar{S}\bar{L}}\right\} = \frac{1}{1 + \rho\bar{S}\bar{L}} \quad (14)$$

$$0 < \alpha < \min\left\{\frac{1}{\bar{\mu}\bar{S}\bar{L}}, \frac{1}{\rho\bar{S}\bar{L}}\right\} = \frac{1}{\rho\bar{S}\bar{L}} \quad (15)$$

since  $1/\rho < 1/\bar{\mu}$ .

Comparing the condition for  $\gamma$  to the stability condition of REM in Section 2, we see that (14) is stricter by an approximate factor of two.

We summarize results of this section in the following theorem, which has already been proven above.

*Theorem 1:* The M-REM algorithm converges to the solution of the primal problem  $(p^* q^*)$ , from any initial condition satisfying  $p(0) \geq 0, b(0) \geq 0, m_s \leq x_s(0) \leq M_s$ , if the step sizes are chosen as in (14)-(15), and the utility functions satisfy the condition (12).

## V. CONCLUSIONS AND FUTURE WORK

In this paper, we proposed a new congestion control algorithm, M-REM, which takes into account the current value of buffer length when calculating the link price. We showed that M-REM is globally asymptotically stable, while preserving the decentralized nature of REM. We are currently investigating stability properties of the algorithm under asynchronous updates.

Future research will be directed towards studying the stability of M-REM in the presence of delay and uncertainty, and comparing its performance with REM, RED [13], and several other Internet congestion control algorithms.

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