"H^∞−robustness of adaptive filters against measurement noise and parameter drift"

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- Consider a scalar sequence $y(t)$ that obeys

$$y(t) \;=\; \phi^T(t)\theta + v(t),\ t = 0, 1, ...$$

where $\theta \in \mathcal{R}^n$ is an unknown parameter vector, $\phi(t) \in \mathcal{R}^n$ is the regressors vector, $v(t) \in \mathcal{R}$ is an unknown disturbance.

- Objective: Estimate the unperturbed output

$$z(t) = \phi^T(t)\theta$$

using the measurements $y(i)$, $0 \le i \le t$.

- If a probabilistic description of $v(t)$ is available, we can use mean-square estimation.

- Alternative H^∞ filtering: Find a causal filter with input y and output $\hat{z}$ such that $J(\bar{\gamma}) \le 0$, where

$$J(\bar{\gamma}) = \sum_{i=0}^{T-1} \left[(z(i) - \hat{z}(i))^2 - \bar{\gamma}^2 v(i)^2 \right] - \bar{\gamma}^2 |\theta - \hat{\theta}_0|^2_{P_0^{-1}}$$

$\hat{\theta}_0$ is an initial guess for θ, and $P_0^{-1} > 0$.

- Consider the following system:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bw_k, \quad k \in [0, N-1] \\
y_k &= C_2 x_k + n_k, \quad k \in [0, N]
\end{aligned}
$$

with

$$J = ||C_1(x_k - \hat{x}_k)||_2^2 - ||w_k||_2^2 - ||n_k||_2^2 - ||x_0 - \hat{x}_0||^2_{R_0^{-1}}.$$

- Objective: Find an estimate $\hat{x}_k$ for x_k based on $\{y_i,\ 1 \leq i \leq k\}$ such that $J < 0$.

- Theorem 2. (I. Yaesh and U. Shaked)

  An estimate $\hat{x}_k$ which achieves $J < 0$ exists iff there exists a solution $\Sigma_k > 0$ to:

$$
\begin{aligned}
\Sigma_{k+1}^{-1} &= M_{k+1}^{-1} + C_2^T C_2 - C_1^T C_1, \quad \Sigma_0 = R_0 \\
M_{k+1} &= A\Sigma_k A^T + BB^T
\end{aligned}
$$

  If such a $\Sigma_k > 0$ exists, then one filter is given by:

$$
\begin{aligned}
\hat{x}_{k+1} &= A\hat{x}_k + K(y_{k+1} - C_2 A\hat{x}_{k+1}),\ \hat{x}_0 \\
K &= M_{k+1} C_2^T (I + C_2 M_{k+1} C_2^T)^{-1}.
\end{aligned}
$$

- Make the correspondence

$$\begin{aligned} \theta_{t+1} &= \theta_t, & A = I, \quad B = 0 \\ y_t &= \phi_t^T \theta + v_t, & C_2 = \phi_t^T, \quad D = 1 \end{aligned}$$

to obtain the solution as:

$$\begin{aligned} \hat{\theta}_t &= \hat{\theta}_{t-1} + K_t[y_t - \phi_t^T \hat{\theta}_{t-1}], \quad \hat{z}_t = \phi_t^T \hat{\theta}_t \\ K_t &= \frac{P_t \phi_t}{1 + \phi_t^T P_t \phi_t} \qquad (1) \\ P_{t+1} &= \left[P_t^{-1} + \phi_t \phi_t^T (1 - 1/\bar{\gamma}^2) \right]^{-1}, \; P_0 > 0 \end{aligned}$$

provided that $P_t > 0$.

- 1. If $\bar{\gamma} \geq 1$, then always $P_t > 0$.

  2. $\bar{\gamma} = 1$, and $P_0 = \mu I \Rightarrow$ (NLMS) estimator.

  3. If the regressors sequence $\{\phi_i\}$ is exciting, i.e.,

  $$\lim_{T \to \infty} \sum_{i=0}^{T-1} \phi_i^T \phi_i = \infty$$

  then $\bar{\gamma} = 1$ is the minimum achievable d.a.l. with $T = \infty$; see (Hassibi et al. 1996a).

  4. The simplistic estimator $\hat{z}_t = y_t$ also achieves $\bar{\gamma} = 1$, but it has the worst H_2 performance among all the estimators yielding $J(1) \leq 0$.

- Introduce the following dynamical model

$$\begin{aligned} \theta_{t+1} &= \theta_t + w_t \\ y_t &= \phi_t^T \theta_t + v_t \\ z_t &= \phi_t^T \theta_t \end{aligned}$$

  and the performance index

$$\begin{aligned} J_\infty(\bar{\gamma}) &= \sum_{i=0}^{T-1} \left[(z_i - \hat{z}_i)^2 - \bar{\gamma}^2 \frac{|w_i|^2}{\bar{q}} - \bar{\gamma}^2 v_i^2 \right] \\ &\quad -\bar{\gamma}^2 |\theta_0 - \hat{\theta}_0|^2_{P_0^{-1}}, \quad \bar{q} > 0. \end{aligned}$$

- Objective: Find the conditions ensuring that the filters of the form (1) achieves $J_\infty(\bar{\gamma}) \le 0$ when

  1. *(NLMS)* : $P_t = \mu I$, $\mu > 0$, $\forall t \ge 0$.

  2. *Central H^∞ filter* : P_t satisfies

$$P_{t+1} = \left[P_t^{-1} + \phi_t \phi_t^T (1 - 1/\gamma^2) \right]^{-1} + qI.$$

  3. *Kalman filter* : P_t satisfies

$$P_{t+1} = \left[P_t^{-1} + \phi_t \phi_t^T \right]^{-1} + qI.$$

- Note that γ and q are design parameters and when $\gamma = \bar{\gamma}$ and $q = \bar{q}$, filters are said to be tuned.

- Theorem 1: Filter (1) achieves $J_\infty(\bar{\gamma}) \le 0$, iff there exists a solution $\Pi_t > 0$ to:

$$\Pi_{t+1} = A_t \Pi_t A_t^T + \bar{q} I + K_t K_t^T + \frac{S_t \phi_t \phi_t^T S_t^T}{\delta_t} \quad (2)$$

$$\begin{aligned} A_t &= I - K_t \phi_t^T, \qquad S_t = A_t \Pi_t A_t^T + K_t K_t^T \\ \delta_t &= \bar{\gamma}^2 - \phi_t^T S_t \phi_t, \quad \Pi_0 = P_0 \end{aligned}$$

such that $\delta_t > 0$, $\forall t \in [0, T-1]$.

- Proof: Let $x_t = \theta_t - \widehat{\theta}_{t-1}$, $e_t = z_t - \widehat{z}_t$ and $d_t = [w_t^T \ v_t^T]^T$. Then, it follows that

$$\begin{aligned} x_{t+1} &= A_t x_t + [I \ \ -K_t] d_t \\ e_t &= \phi_t^T A_t x_t + [0 \ \ -\phi_t^T K_t] d_t. \end{aligned} \quad (3)$$

Now, invoke the following lemma.

- Lemma (I. Yaesh and U. Shaked): Consider

$$x_{k+1} = A x_k + B w_k, \ y_k = C x_k + D w_k.$$

Then, $J = ||y_k||_2^2 - ||w_k||_2^2 - ||x_0||^2_{R_0^{-1}} \le 0$ iff there exist a solution $M_k > 0$ to

$$\begin{aligned} M_{k+1} &= A M_k A^T + B B^T + [A M_k C^T + B D^T] \\ & \quad [I - C M_k C^T - D D^T]^{-1} [C M_k A^T + D B^T] \end{aligned}$$

with $M_0 = R_0^{-1}$ so that $I - C M_k C^T - D D^T > 0$.

- Theorem 2: For any given $\bar{\gamma} > 0$ and $\mu > 0$, there exists a sufficiently large T and a persistently exciting sequence $\{\phi_i\}$ such that NLMS does not guarantee $J_\infty(\bar{\gamma}) \leq 0$.

- Proof. The condition $\delta_t > 0$ becomes:

$$\phi_t^T \Pi_t \phi_t < \bar{\gamma}^2 (1 + \mu \phi_t^T \phi_t)^2 - (\mu \phi_t^T \phi_t)^2.$$

Let $\bar{n} := \lceil (\mu/\bar{q})(4\bar{\gamma}^2 - 1)^+ \rceil$ and

$$\phi_t = 0, \quad t \in [\bar{t}, \bar{t} + \bar{n} - 1], \quad \phi_{\bar{t}+\bar{n}}^T \phi_{\bar{t}+\bar{n}} = 1/\mu.$$

Since $\Pi_{\bar{t}+\bar{n}} \geq \bar{n}\bar{q}I$, we have

$$\begin{aligned} |\phi_{\bar{t}+\bar{n}}|^2_{\Pi_{\bar{t}+\bar{n}}} \geq \frac{\bar{n}\bar{q}}{\mu} \quad &\geq \quad 4\bar{\gamma}^2 - 1 \\ &= \quad \bar{\gamma}^2 (1 + \mu |\phi_{\bar{t}+\bar{n}}|)^2 - \mu^2 |\phi_{\bar{t}+\bar{n}}|^2 \end{aligned}$$

which violates $\delta_t > 0$.

- If $\bar{\gamma} = \gamma$ and $\bar{q} = q$, then Π_t coincides with P_t and $\delta_t > 0$ condition reduces to

$$P_t^{-1} + \phi_t \phi_t^T (1 - 1/\bar{\gamma}^2) > 0, \quad \forall t \in [0, T-1] \qquad (4)$$

which is always satisfied for $\bar{\gamma} \geq 1$.

- Theorem 3: For any $\bar{\gamma} \geq 1$ and $T > 0$, the tuned H^∞ filter guarantees $J_\infty(\bar{\gamma}) \leq 0$

- Theorem 4: For any $\bar{\gamma} < 1$ and any sequence with

$$\lim_{T \to \infty} \sum_{i=0}^{T-1} i^{-\epsilon} \phi_i^T \phi_i = \infty,$$

$\exists$ a sufficiently large interval $[0, T-1]$ such that the tuned H^∞ filter does not guarantee $J_\infty(\bar{\gamma}) \leq 0$.

- Proof. As long as $P_t > 0$, (4) is equivalent to

$$\phi_t^T P_t \phi_t < \bar{\gamma}^2 / (1 - \bar{\gamma}^2).$$

Since $\bar{\gamma} < 1$, $\phi_t^T P_t \phi_t \geq q t \phi_t^T \phi_t$. From $\epsilon-$excitation, $\exists$ a large $\bar{t}$ such that $\phi_{\bar{t}}^T \phi_{\bar{t}} > \bar{\gamma}^2 / (\bar{q}\bar{t}(1 - \bar{\gamma}^2))$. This implies $\phi_t^T P_t \phi_t > \bar{\gamma}^2 / (1 - \bar{\gamma}^2)$, a contradiction.

- Theorem 3 and 4 implies the minimum achievable d.a.l. $\bar{\gamma} = 1$ when $T = \infty$. When $\bar{\gamma} = 1$ $P_t = P_0 + \bar{q}tI$ and the tuned H^∞ filter becomes

$$\hat{z}_t = \frac{\phi_t^T \hat{\theta}_{t-1} + (\phi_t^T P_0 \phi_t + \bar{q}t\phi_t^T \phi_t) y_t}{1 + \phi_t^T P_0 \phi_t + \bar{q}t\phi_t^T \phi_t}$$

  which asymptotically approaches to the simplistic estimator whenever $\phi_t \neq 0$ and $\lim_{t\to\infty} t\phi_t^T \phi_t = \infty$.

- When ϕ_t is uniformly persistently exciting, i.e.,

$$\alpha I < \sum_{i=k}^{k+l-1} \phi_i \phi_i^T < \beta I$$

  for some α, β, l, $\forall k \geq 0$, then the central H^∞ filter with $\gamma > 1$ is exponentially stable. This implies parameter convergence (exponentially) when $w_t, v_t \in \mathcal{L}_2$.

- Theorem 6: Assume ϕ_t is uniformly persistently exciting and consider the Kalman filter with$q > 0$. Then:

  1. P_t is bounded.

  2. The closed-loop filter matrix $I - K_t\phi_t^T$ is exponentially stable.

  3. The filter ensures a finite d.a.l.

- Theorem 7: If $q \geq \bar{q}$, then Kalman filter guarantees $J_\infty(2) \leq 0$, for any $T > 0$.

- Proof. The tuned case $q = \bar{q}$ is proven in (Hassibi 96). If $q > \bar{q}$, then $J_\infty^q(\bar{\gamma}) > J_\infty^{\bar{q}}(\bar{\gamma})$.

- Kalman filter has better robustness properties than NLMS. And, $H_\infty-$filter is better than Kalman filter only for $\bar{\gamma} < 2$.

- H^2 design: Consider the error system (3), and find a filter gain K_t such that J_2 is minimized, where w_t and v_t are independent white noises with variances $\bar{q}I$ and 1, and

$$J_2 = \frac{1}{T} E \left[\sum_{i=0}^{T-1} e_i^2 \right].$$

J_2 can be written as:

$$J_2 = \frac{1}{T} \sum_{i=0}^{T-1} \phi_t^T [A_t \bar{P}_t A_t^T + K_t K_t^T] \phi_t$$

where $\bar{P}_t$ satisfies:

$$\bar{P}_{t+1} = A_t \bar{P}_t A_t^T + \bar{q}I + K_t K_t^T, \; \bar{P}_0 = P_0.$$

J_2 is minimized by the tuned Kalman filter.

- The mixed H^2/H^∞ problem is to find a filter such that J_2 is minimized while keeping $J_\infty(\bar{\gamma}) \leq 0$, for some fixed $\bar{\gamma} > 0$.

- The time-invariant infinite horizon H^2/H^∞ problem is studied in (Halder et al 96) and (Sznaier et al 96). The parameter estimation problem is more complicated because ϕ_t is time-varying.

- Note that for any filter K_t with $J_\infty(\bar{\gamma}) \leq 0$, we have

$$J_2 = \frac{1}{T}\sum_{i=0}^{T-1} \phi_t^T [\bar{P}_{t+1} - \bar{q}I]\phi_t \leq \bar{J}_2$$

where

$$\bar{J}_2 := \frac{1}{T}\sum_{i=0}^{T-1} \phi_t^T [\Pi_{t+1} - \bar{q}I]\phi_t.$$

- Theorem 8: Among all filters guaranteeing $J_\infty(\bar{\gamma}) \leq 0$ for a given $\bar{\gamma} > 0$, the tuned H^∞ filter minimizes $\bar{J}_2$. Moreover, $\bar{J}_2$ is non-increasing with $\bar{\gamma}$.

- Proof. (2) can be written as:

$$\Pi_{t+1} - \bar{q}I = \left[S_t^{-1} - \phi_t\phi_t^T/\bar{\gamma}^2\right]^{-1},$$

$$S_t = [\Pi_t^{-1} + \phi_t\phi_t^T]^{-1} + N_t[1 + \phi_t^T\Pi_t\phi_t]N_t^T$$

$$N_t = K_t - \frac{\Pi_t\phi_t}{1 + \phi_t^T\Pi_t\phi_t}$$

$N_t = 0$ minimizes $\bar{J}_2$. The second part follows from the monotonicity of Π_t as a function of $\bar{\gamma}$.