University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 459: Communications I

Fall 2004

Problem Set 7

Random Processes through LTI Systems, Power Spectral Density, Bandpass Random Processes

Issued: Thursday, Oct. 28th. **Due:** Thursday, Nov. 4th (beginning of lecture).

Reading from Lathi: Chapter 11, Sections 11.1-11.5.

Reading from Haykin: Chapter 4, Sections 4.11–4.14.

Announcement: The second Mid-Semester Exam will be held on Tuesday, November 9th, from 5:00pm to 7:00pm in 165 Everitt. The exam will cover all material from the beginning of the term *up to and including* the lecture on Thursday, November 4th. The corresponding material includes (the material covered in the first Mid-Semester Exam and) Problem Sets 4 through 7 and
(i) **Lathi**: Chapters 5, 10, and 11 (excluding Section 11.6), OR
(ii) **Haykin (3rd Edition)**: Chapters 3 and 4 (excluding Section 4.15).
For the exam, you can bring *two* 8.5×11-inch double-sided sheets of *handwritten* notes. Calculators are allowed but will not be necessary.

Problem 7.1 (Optional)

Problem 11.3.1 from Lathi, p. 528.

Problem 7.2

A random process $X(t)$ is defined as $X(t) = A\cos(\omega_c t)$, where ω_c is a constant and A is a Gaussian random variable with zero mean and variance σ_A^2. This random process is applied to an ideal integrator producing the output $Y(t) = \int_0^t X(\tau)d\tau$.

(a) Determine the first order pdf of the output $Y(t)$ (i.e., find $f_{Y(t_1)}(y)$ for any time instant $t_1 > 0$.

(b) Is $Y(t)$ wide-sense stationary (WSS)? Is $Y(t)$ strict-sense stationary (SSS)?

Problem 7.3 (Optional)

Let $X[n]$ be a discrete-time random process defined for $n = ..., -1, 0, 1, 2, ...$. Each $X[i]$ is independent from all other samples $X[j]$, $j \neq i$, and has pdf $f_{X[i]}(x)$ that is uniform in $[-1, 1]$. Define the random process $Y[n]$ to be

$$Y[n] = \frac{2}{3}X[n] + \frac{1}{3}X[n-1]\ .$$

(a) Are $Y[n_1]$ and $Y[n_2]$ for $n_1 \neq n_2$ independent?

(b) Find $E[Y[n]]$ and $R_{YY}[n_1, n_2]$.

(c) Is $Y[n]$ a wide-sense stationary random process?

(d) Find the LMMSE of $Y[5]$ given $Y[4] = y$.

Problem 7.4

Problem 11.2.1 from Lathi, p. 526.

Problem 7.5

A zero-mean Gaussian random process $X(t)$ has power spectral density

$$S_{XX}(\omega) = \frac{4}{1+\omega^2}\,, \qquad -\infty < \omega < +\infty\,.$$

(a) Determine $R_{XX}(\tau)$, the autocorrelation function of the random process $X(t)$.

(b) The random process $X(t)$ is passed through a stable LTI system with frequency response

$$H(\omega) = \begin{cases} 1\,, & |\omega| < 2\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Determine the average power $E[Y^2(t)]$ of the output random process $Y(t)$.

Problem 7.6

(a) Let random processes $X(t)$ and $Y(t)$ be the input and output respectively of a stable LTI system with frequency response $H(\omega)$. Assume $X(t)$ is wide-sense stationary and define random process $Z(t)$ to be

$$Z(t) = Y(t) - X(t)\,.$$

Find $S_{ZZ}(\omega)$, the power spectral density of $Z(t)$, in terms of $H(\omega)$ and $S_{XX}(\omega)$.

(b) The input voltage $X(t)$ to a stable LTI filter with frequency response

$$H(\omega) = \frac{2}{2+j\omega}$$

can be modeled as a wide-sense stationary random process with zero mean and autocorrelation function $R_{XX}(\tau) = 2e^{-|\tau|}$. Find $S_{ZZ}(\omega)$, the power spectral density of random process $Z(t) = Y(t) - X(t)$, where $Y(t)$ is the output of the filter.

Problem 7.7

A wide-sense stationary random process $X(t)$ with autocorrelation function $R_{XX}(\tau) = e^{-|\tau|}$ is processed by a stable LTI system with real-valued impulse response $h(t)$.

(a) For this part, assume that the output of the filter $Y(t)$ is a wide-sense stationary random process with autocorrelation function $R_{YY}(\tau) = 3e^{-3|\tau|}$.

(i) Find $|H(\omega)|$, the magnitude of the frequency response of the filter.

(ii) Suppose that $h(t)$ is causal. Find a possible impulse response $h(t)$ for the LTI system. Is your answer unique?

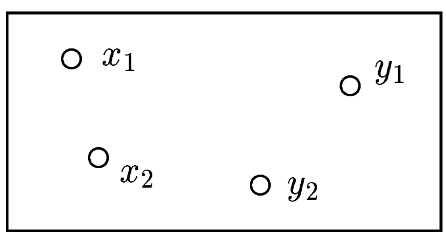
(b) Suppose that the LTI system is known to be stable and that

$$R_{YX}(\tau) = e^{-\tau}u(\tau) - 2e^{-2\tau}u(\tau) + e^{-3\tau}u(\tau) \ .$$

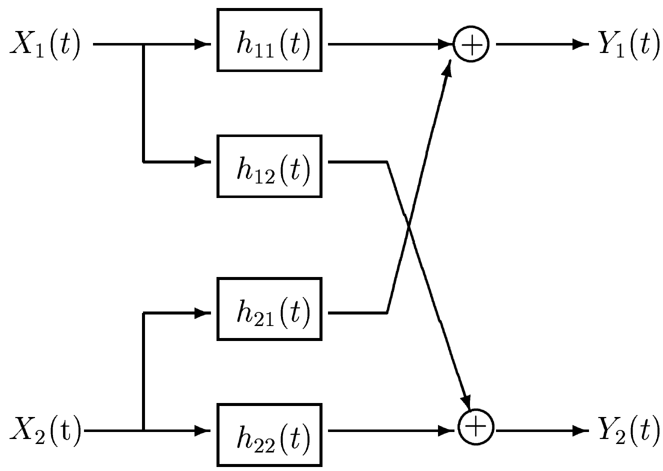
Find a possible impulse response $h(t)$. Is your answer unique?

Problem 7.8

In a variety of real communication situations two (or more) uncorrelated sources are received through channels or systems with crosstalk. Since signals interfere with each other, it is essential that they are separated at the receiving end. One such scenario is shown below, where we have two sources, denoted by x_1 and x_2, and two receivers, denoted by y_1 and y_2.



The above scenario is modeled in terms of LTI systems as shown below. The impulse response from source i to receiver j is denoted by $h_{ij}(t)$.



Approaches for recovering the signals from sources x_1 and x_2 usually involve estimating the impulse responses $h_{ij}(t)$. In this problem we investigate such techniques; to simplify the problem we assume that $h_{11}(t) = h_{22}(t) = \delta(t)$. We also assume that $h_{12}(t)$ and $h_{21}(t)$ are stable and causal and that the sources can be modeled by uncorrelated wide-sense stationary random processes $X_1(t)$ and $X_2(t)$ (with zero mean and known autocorrelation functions $R_{X_1X_1}(\tau)$ and $R_{X_2X_2}(\tau)$).

(a) Show that $Y_1(t)$ and $Y_2(t)$ are jointly wide-sense stationary and determine $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ and $R_{Y_1Y_2}(\tau)$ in terms of $R_{X_1X_1}(\tau)$, $R_{X_2X_2}(\tau)$, $h_{12}(t)$ and $h_{21}(t)$.

(b) Suppose that $h_{12}(t) = h_{21}(t) = h(t)$ and that you can measure *only one* of $R_{Y_1Y_1}(\tau)$, $R_{Y_2Y_2}(\tau)$ or $R_{Y_1Y_2}(\tau)$. Which one would be most helpful in determining $h_{12}(t)$?