Robust Control of Heterogeneous Networks

From slides of Prof. Glenn Vinnicombe “Robust Control of Heterogeneous Networks”

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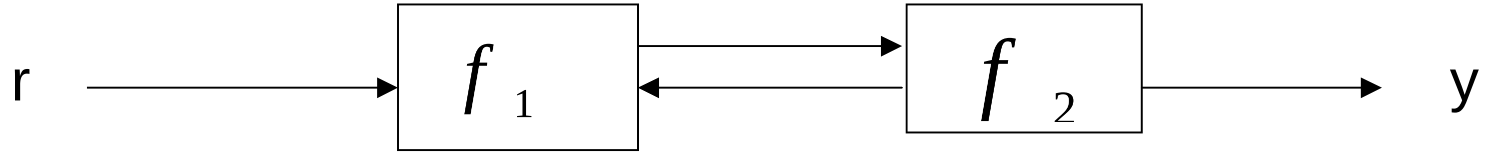
Introduction

For a system like following (assume f_i and g_i are all stable and SISO), how can we analysis the stability of the system?



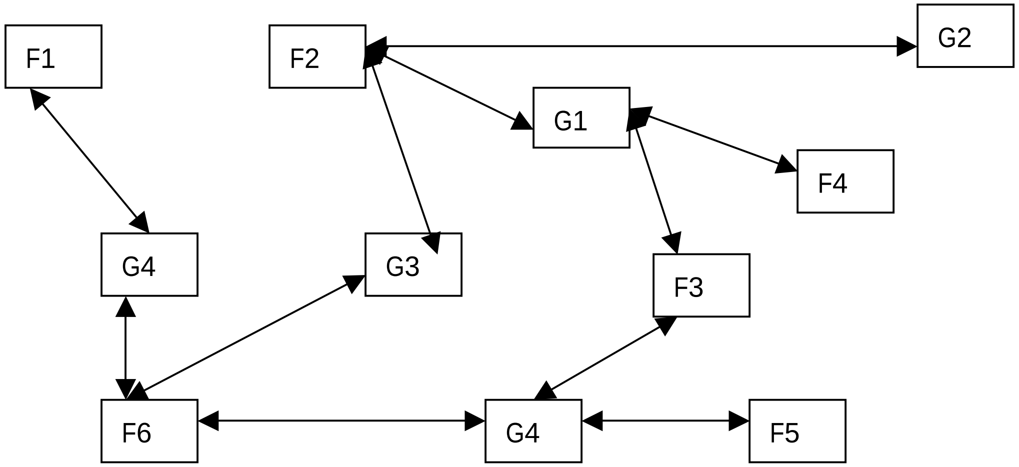
We can write down the transfer function according to the global structure of the system

$$H_{yr} = \frac{f_1 f_2}{1 - f_1 f_2}$$

then

all zeros of $1 - f_1 f_2$ are at left-half plane $\Leftrightarrow$ the system is stable.

But if the system is as following:



It will contain thousands of nodes (for example “Internet”).

It will be very hard to express the transfer function for the whole system.

Then what can we do if we want to analysis the system and then build a controller?

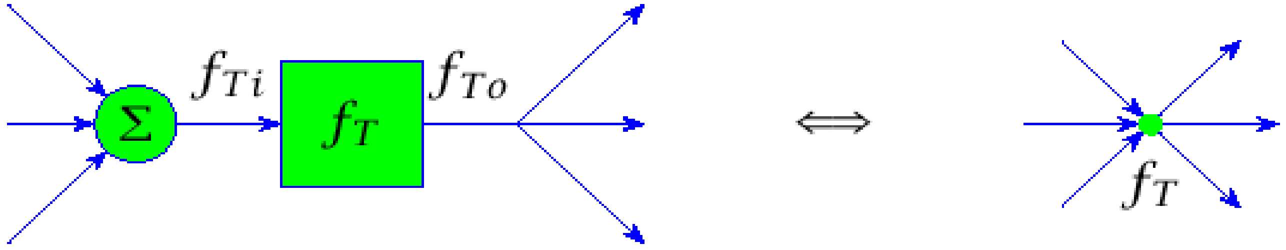
Two problem:

Is it possible to build stable and robust large scale feedback interconnections of linear dynamical systems?

Is it possible to get the global stability from the locally condition?

Yes, but only limited results.

Signal Flow Graphs (Mason '53)



Assume linearity, and that Laplace transforms have been taken everywhere, so

$$f_{To}(s) = f_T(s) f_{Ti}(s)$$

Using Signal Flow Graphs. We consider the network as following

There are two kinds of nodes in the network, we can think f is the plant and g is the controller. Now assume $f_{lo}(s) = f_l(s) f_{li}(s) : l = 1,2$, $g_{lo}(s) = g_l(s) g_{li}(s) : l = 1,2,3,4$ and all of them are stable.

$$\begin{cases} g_{1o} = g_1 f_{1o} \\ g_{2o} = g_2 (f_{1o} + f_{2o}) \\ g_{3o} = g_3 (f_{1o} + f_{2o}) \\ g_{4o} = g_4 f_{2o} \end{cases} \Leftrightarrow \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{3o} \\ g_{4o} \end{bmatrix} = \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & g_3 & \\ & & & g_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix}$$

Also we know that

$$\begin{cases} f_{1i} = g_{1o} + g_{2o} + g_{3o} \\ f_{2i} = g_{2o} + g_{3o} + g_{4o} \end{cases} \Leftrightarrow \begin{bmatrix} f_{1i} \\ f_{2i} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_{1o} \\ g_{2o} \\ g_{3o} \\ g_{4o} \end{bmatrix}$$

So we have

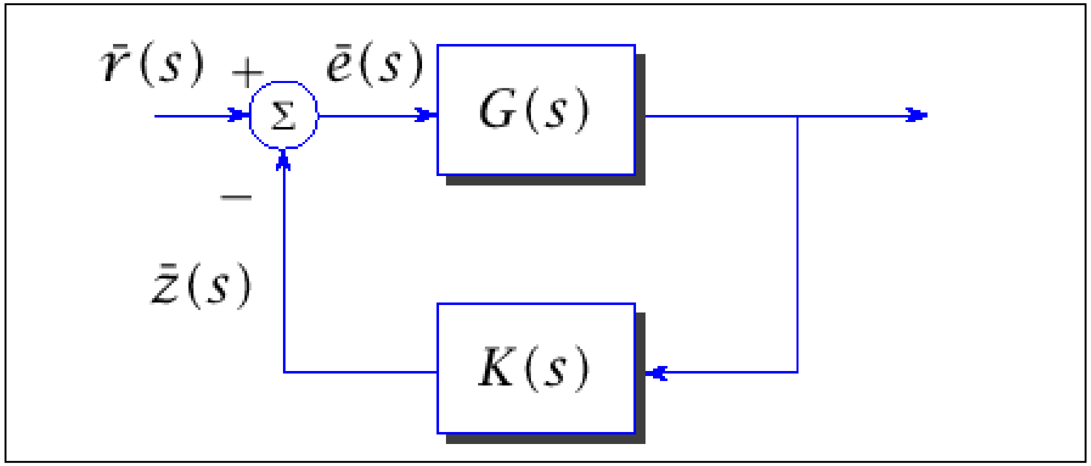
$$\begin{bmatrix} f_{1i} \\ f_{2i} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{R} \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & g_3 & \\ & & & g_4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R^T} \begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix}$$

then we get the return ratio $\begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix} \rightarrow \begin{bmatrix} f_{1o} \\ f_{2o} \end{bmatrix}$ is

$$\underbrace{\begin{bmatrix} f_1 & \\ & f_2 \end{bmatrix}}_{F(s)} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & g_3 & \\ & & & g_4 \end{bmatrix}}_{G(s)} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R^T}$$

explanation of “return ratio”

for a general feedback system as following



The “return ratio” is $G(s)F(s)$. If G, F are both stable, we have:

Closed-loop system is stable⇔zeros of $\det(I+GF) \notin C_+$

So we have

closed-loop system is stable $\Leftrightarrow$ ***zeros of*** $\det(I - FRGR^T) \notin C_+$

(because f_i ***and*** g_i ***are all stable)***

According to the “generalized Nyquist stability criterion”, this equals

Closed-loop stable $\Leftrightarrow \{\lambda_i(F(j\omega)RG(j\omega)R^T) : \omega \in \Re\}$

do not encircle +1

This is a very cool result and the proof will be given later.

Now we have a way to check the stability of above network system, but this is still a global condition, require the knowledge of the structure of the whole system.

What we will do next is to change the global condition to locally verifiable condition.

Concept of convex hull

$$Co(a_1, a_2, a_3) = \mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 \qquad 0 \le \mu_i \le 1, \mu_1 + \mu_2 + \mu_3 = 1$$

1. *If the g is real and positive.*

We have LEMMA:

Let $Q = Q^* \geq 0, f_i \in C$, then

$$\sigma(diag(f_1, f_2, \cdots, f_n)Q) \subset Co(0 \cup \{f_i\})\rho(Q) \qquad (*1)$$

According to this lemma, we can get:

$$\sigma(FRGR^T) \subset Co(0 \cup \{f_i d_i g_j e_j : i, j, R_{ij} \neq 0\}) \qquad (*2)$$

d_i is the number of g connected to f_i; and e_i is the number of f connected to $g_i \Rightarrow$

Closed-loop stable $\Leftrightarrow 1 \notin Co(0 \cup \{f_i d_i g_j e_j : i, j, R_{ij} \neq 0\})$.

2. If g is dynamical

$$\sigma(R^T FRG) \subset \rho(R^T R) \cdot Co\{ f_j \left(Co\left\{ \pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots \right\} \right)^2 \} \qquad (*3)$$

Then we have

Closed-loop stable $\Leftrightarrow 1 \notin \rho(R^T R) \cdot Co\{ f_j \left(Co\left\{ \pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots \right\} \right)^2 \}$

If we have $\rho(R^T R) \leq 1$, we get

Closed-loop stable $\Leftrightarrow 1 \notin Co\{ f_j \left(Co\left\{ \pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots \right\} \right)^2 \}$

Now we can see that they are both **locally verifiable conditions**. What we need to is to construct the corresponding convex hull. It will be easy finished by MATLAB.

Therefore whether how many nodes there are in the network, we can construct the corresponding convex hull to judge the stability according to the locally verifiable conditions. The scale of the network will not be a problem. Also when some nodes added or deleted, it is also very easy to adjust the convex hull, we just need to add or deleted corresponding nodes of the figure.

Now we give the proof of (*1) (*2) and (*3)

Proof of (*1) $\sigma(diag(f_1, f_2, \cdots, f_n)Q) \subset Co(0 \cup \{f_i\})\rho(Q)$

for $Q = Q^* \geq 0, f_i \in C \ \Rightarrow$

$$\sigma(diag(f_1, f_2, \cdots, f_n)Q) = \sigma(Q^{1/2} diag(f_1, f_2, \cdots, f_n)Q^{1/2})$$

$$\subset \{v^* Q^{1/2} diag(f_1, f_2, \cdots, f_n)Q^{1/2} v : \|v\| = 1\}$$

since $\left\|Q^{1/2} v\right\| \leq \left\|Q^{1/2}\right\| = \sqrt{\rho(Q)} \Rightarrow Q^{1/2} v \subset \{\sqrt{\rho(Q)} w : \|w\| \leq 1\} \Rightarrow$

$$\{v^* Q^{1/2} diag(f_1, f_2, \cdots, f_n)Q^{1/2} v : \|v\| = 1\}$$

$$\subset \rho(Q)\{\sum_i |w_i|^2 f_i : \sum_i |w_i|^2 \leq 1\}$$

$$\subset \rho(Q)\{(\sum_i |w_i|^2 f_i) + (1 - \sum_i |w_i|^2)\cdot 0 : \sum_i |w_i|^2 \leq 1\}$$

$$\subset \rho(Q) Co(0 \cup \{f_i\})$$

*Proof of (*2)* $\sigma(FRGR^T) \subset Co(\{ f_i d_i g_i e_i : i, j, R_{ij} \neq 0\})$

1. Suppose $R_{m\times n}$;

2. let $d_i = \sum_{l=1}^{n} r_{il}$, the i’th row sum of $R_{m\times n}$, it means how many $g_{\cdot}$ directly connect to f_i.

3. let $e_j = \sum_{l=1}^{m} r_{lj}$, the i’th row sum of R^T, it means how many $f_{\cdot}$ directly connect to g_j;

we know that

$$RGR^T = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{bmatrix} \begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{bmatrix} \begin{bmatrix} r_{11} & \cdots & r_{m1} \\ \vdots & \ddots & \vdots \\ r_{1n} & \cdots & r_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}g_1 & \cdots & r_{1n}g_n \\ \vdots & \ddots & \vdots \\ r_{m1}g_1 & \cdots & r_{mn}g_n \end{bmatrix} \begin{bmatrix} r_{11} & \cdots & r_{m1} \\ \vdots & \ddots & \vdots \\ r_{1n} & \cdots & r_{mn} \end{bmatrix}$$

If define $(RGR^T)_{ij} = a_{ij} = \sum_{l=0}^{n} r_{il} g_l r_{jl}$

So we know that the sum of the row is $\sum_{j=0}^{m} \sum_{l=0}^{n} r_{il} g_l r_{jl}$

Then define the row sum is S_i

$$S_i = \sum_{j=0}^{m} \sum_{l=0}^{n} r_{il} g_l r_{jl} = \sum_{l=0}^{n} \sum_{j=0}^{m} r_{il} g_l r_{jl} = \sum_{l=0}^{n} r_{il} g_l \sum_{j=0}^{m} r_{jl}$$

$$= \sum_{l=0}^{n} r_{il} g_l e_l = \sum_{j=0}^{n} r_{ij} g_j e_j$$

Now we get

$$FRGR^{T} = F'Q' = \begin{bmatrix} f_1S_1 & & \\ & \ddots & \\ & & f_mS_m \end{bmatrix} \begin{bmatrix} \frac{a_{11}}{S_1} & \cdots & \frac{a_{1m}}{S_1} \\ \vdots & \ddots & \vdots \\ \frac{a_{m1}}{S_m} & \cdots & \frac{a_{11}}{S_m} \end{bmatrix}$$

and we can prove that $\rho(Q') \leq \max$ {row sum of Q'}

$\Rightarrow \rho(Q') \leq \max$ {row sum of Q'}=1

Proof of $\rho(Q') \le \max$ ***{row sum of*** Q' ***}***

First we have three well-know result

1. $\rho(Q') \le \|Q'\|$ for any matrix norm $\|\cdot\|$
2. maximum row sum $\max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} |q'_{ij}| = \|Q'\|$ is a norm ("Matrix analysis" p. 295)
3. if $Q'' \ge Q' \ge 0 \Rightarrow \rho(Q'') \ge \rho(Q')$

So we first choose $Q'' \geq Q' \geq 0$ and let $\sum_{j=1}^{n} \left| q''_{ij} \right| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| q'_{ij} \right| = \alpha$ for $i = 1, 2, \cdots, n$. So the row sum of Q'' is a constant and equals the maximum row sum of Q'.

So it is easy to check that α is a eigenvalue of Q'' with eigenvector $x = [1, 1, \cdots, 1]^T \Rightarrow \rho(Q'') \geq \alpha$

We know that from 1 and 2 that $\rho(Q'') \leq \left\| Q'' \right\| = \alpha$

$\Rightarrow \alpha = \rho(Q'')$

And also from 3 we have $Q'' \geq Q' \geq 0 \Rightarrow \rho(Q'') \geq \rho(Q')$

then

$$\Rightarrow \alpha = \rho(Q'') \geq \rho(Q')$$

with $\rho(Q') \leq \max$ ***{row sum of*** Q' ***}=1 and Lemma (*1), we can get***

$$\sigma(FRGR^T) \subset Co(0 \cup \{f_i S_i\})$$

now prove $Co(0\cap\{f_iS_i\})\subset Co\{0\cup f_id_ig_je_j : j=1,2,\cdots,m,r_{ij}\neq 0\}$

We have get $f_iS_i = f_i\sum_{j=0}^{n} r_{ij}g_je_j = \sum_{j=0}^{n} f_ir_{ij}g_je_j$ and $\sum_{j=0}^{n} r_{ij}=d_i$. Then

$$f_iS_i = \sum_{j=0}^{n}\frac{d_i}{d_i} f_ir_{ij}g_je_j = \sum_{j=0}^{n}\frac{r_{ij}}{d_i}(f_id_ig_je_j) = \sum_{j=0}^{n}\frac{r_{ij}}{d_i}(f_id_ig_je_j)$$

We know that $\sum_{j=0}^{n}\frac{r_{ij}}{d_i}=1 \Rightarrow f_iS_i\in Co\{f_id_ig_je_j : j=1,2,\cdots,m,r_{ij}\neq 0\}$

$$\sigma(FRGR^T)\subset Co(0\cup\{f_iS_i\})\subset Co\{0\cup\{f_id_ig_je_j,r_{ij}\neq 0\}\}$$

Therefore g_i real we can get the following result:

$1 \notin Co(\{0 \cup \{f(j\omega)_i d_i g_j e_j, r_{ij} \neq 0\}\}) \Rightarrow$ The system is stable

Note: This is a locally verifiable condition

*Proof of (*3)(for dynamic* g_i*)*

Let $R \in \Re^{m\times n}$ satisfy , and $G = diag\left(g_1, \cdots, g_n\right)$,

$F = diag\left(f_1, \cdots, f_m\right)$, g_i , $f_i \in C$

$\forall i$ then

$$\sigma\left(R^T FRG\right) \subset \rho(R^T R) \cdot Co\{ f_j\left(Co\left\{\pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots\right\}\right)^2, j = 1,2,\cdots\}$$

Proof.

let $G^{1/2} = diag\left(\sqrt{g_1},\cdots,\sqrt{g_n}\right)$ where either value of each square root may be used, then

$$\sigma\left(GR^T FR\right) \subset \left\{v^* G^{1/2} R^T FRG^{1/2} v : v \in C^n, v^* v = 1\right\}$$

$$= \left\{\sum_k f_k v^* G^{1/2} R_{k\bullet}^T R_{k\bullet} G^{1/2} v : v \in C^n, v^* v = 1\right\}$$

$$=\left\{\sum_k f_k\left(v_1^*R_{k1}^T\sqrt{g_1}+v_2^*R_{k2}^T\sqrt{g_2}+\cdots\right)\right.$$

(**)

$$\left.\left(v_1R_{k1}\sqrt{g_1}+v_2R_{k2}\sqrt{g_2}+\cdots\right):v\in C^n, v^*v=1\right\}$$

Next, note that for any $\alpha\in C^n$, since $\Re\left(\alpha_1^*\alpha_2\right)\in\left[-\left|\alpha_1\alpha_2\right|,\left|\alpha_1\alpha_2\right|\right]$

$$\left(\alpha_1^*\sqrt{g_1}+\alpha_2^*\sqrt{g_2}+\cdots\right)\left(\alpha_1\sqrt{g_1}+\alpha_2\sqrt{g_2}+\cdots\right)$$

$$=\left|\alpha_1\right|^2g_1+\left|\alpha_2\right|^2g_2+\cdots 2\Re\left(\alpha_1^*\alpha_2\right)\sqrt{g_1g_2}+\cdots$$

$$\in Co\left\{\left(|\alpha_1|\sqrt{g_1}+|\alpha_2|\sqrt{g_2}+\cdots\right)^2,\left(|\alpha_1|\sqrt{g_1}-|\alpha_2|\sqrt{g_2}+\cdots\right)^2,\cdots\right\}$$

since $\left(|\alpha_1|\sqrt{g_1}+|\alpha_2|\sqrt{g_2}+\cdots\right)\in\left(|\alpha_1|+|\alpha_2|+\right.$

$$\left.\cdots Co\left\{\sqrt{g_1},\sqrt{g_2},\cdots:\alpha_i\neq 0\right\}\right)$$

$$\subset\left(|\alpha_1|+|\alpha_2|+\cdots\right)^2 Co\left\{\left(Co\left\{\sqrt{g_1},\sqrt{g_2},\cdots\right\}\right)^2,\right.$$

$$\left.\left(Co\left\{\sqrt{g_1},-\sqrt{g_2},\cdots\right\}\right)^2,\cdots\right\}$$

$$\subset\left(|\alpha_1|+|\alpha_2|+\cdots\right)^2 Co\left\{\left(Co\left\{\pm\sqrt{g_1},\pm\sqrt{g_2},\cdots\right\}\right)^2\right\}$$

(since each term, e.g. $\left(Co\left\{\sqrt{g_1}, -\sqrt{g_2}, \cdots\right\}\right)^2$,

$\subset \left(Co\left\{\pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots\right\}\right)^2$)

Then from (**)

$$\sigma\left(R^T FRG\right) \subset \left\{\sum_k f_k \left(\left|v_1 R_{k1}\right| + \left|v_2 R_{k2}\right| + \cdots\right)^2 \right.$$

$$\left. \cdot Co\left\{\left(Co\left\{\pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots\right\}\right)^2\right\} : v \in C^n, v^* v = 1\right\}$$

we know that $\sum_k f_k a_k Co\{g_i\} = \sum_k a_k Co\{f_k g_i\}$

Also for fixed k, we can get $Co\{f_k g_i\} \subset Co\{f_j g_i, j = 1,2,\cdots\}$

Then we get $\sum_k a_k Co\{f_k g_i\} \subset \sum_k a_k Co\{f_j g_i, j = 1,2,\cdots\}$

$$\sum_k a_k Co\{f_j g_i, j = 1,2,\cdots\} = (\sum_k a_k) \cdot Co\{f_j g_i, j = 1,2,\cdots\}$$

from the above we can get

$$\sum_k f_k \left(\left|v_1 R_{k1}\right| + \left|v_2 R_{k2}\right| + \cdots\right)^2 \cdot Co\left\{\left(Co\left\{\pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots\right\}\right)^2\right\}$$

$$\subset \left(\sum_k \left(|v_1 R_{k1}| + |v_2 R_{k2}| + \cdots \right)^2 \right) \cdot Co\{ f_j \left(Co\left\{ \pm \sqrt{g_1}, \pm \sqrt{g_2}, \cdots \right\} \right)^2, j = 1,2,\cdots \}$$

we know that for $\forall v_i$, we always can find $v_i' = \dfrac{|v_1 R_{k1}|}{R_{k1}}$ such that

$v_i' R_{k1} = |v_1 R_{k1}|$, also $v'^* v' = 1$

$$\rho(R^T R) = \max\{v'^* R^T R v' : v'^* v' = 1\} = \max\{\sum_k |v_1' R_{k1} + v_2' R_{k2} + \cdots|^2\}$$

$$\geq \sum_k \left(|v_1 R_{k1}| + |v_2 R_{k2}| + \cdots \right)^2 \Rightarrow$$

$$\sigma(R^T F R G) \subset \rho(R^T R) \cdot Co\{ f_j \left(Co\left\{ \pm \sqrt{g_1}, \pm \sqrt{g_2}, \cdots \right\} \right)^2, j = 1,2,\cdots \}$$

Review of the result

1. If the g is real and positive.

Closed-loop stable $\Leftrightarrow 1 \notin Co(0 \cup \{ f_i d_i g_j e_j : i, j, R_{ij} \neq 0 \})$.

2. If g is dynamical

Closed-loop stable $\Leftrightarrow 1 \notin Co\{ f_j \left(Co\left\{ \pm\sqrt{g_1}, \pm\sqrt{g_2}, \cdots \right\} \right)^2 \} : \rho(RR^T) \prec 1$

Both condition are local