Randomized algorithms for quadratic stability of quantized sampled-data systems

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Outline

1. Introduction
2. Problem formulation
3. Probabilistic approach
4. A modified sufficient condition
5. An efficient randomized algorithm
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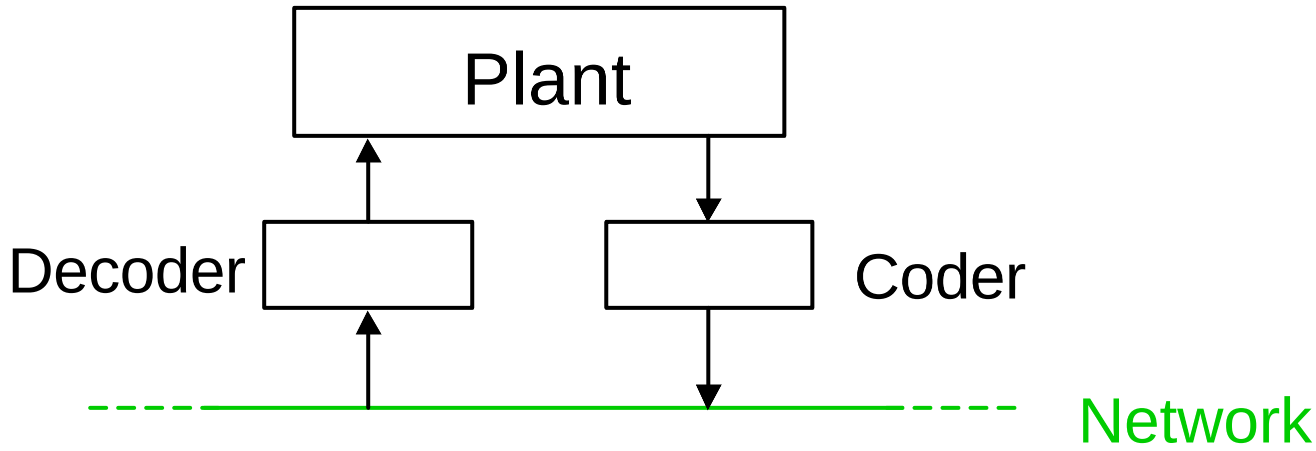
1. Introduction

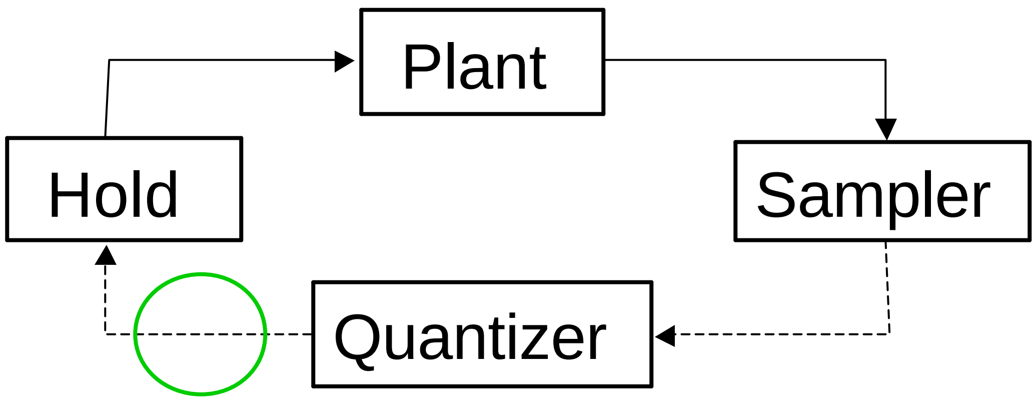
Quantization in control systems



A finite number of bits per message

How small can the data rate be for control purposes?

Sampled-data systems with memoryless quantizers



- Quantizer: memoryless

  a finite number of fixed values

- Asymptotic stability impossible
- How close to the equilibrium do trajectories go?

Sampled-data systems with memoryless quantizers

Analysis/design with analytic bounds on trajectories

- Hou, Michel, and Ye (1997)
- Wong and Brockett (1999)
- Ishii and Francis (2001)
- Ishii and Basar (2002)

Very conservative results

In this work

- Quadratic stability in continuous time
- Probabilistic approach analysis
- Develop an gradient-based algorithm
- Sequentially finds a quadratic Lyapunov function
- A systematic way for realistic bounds

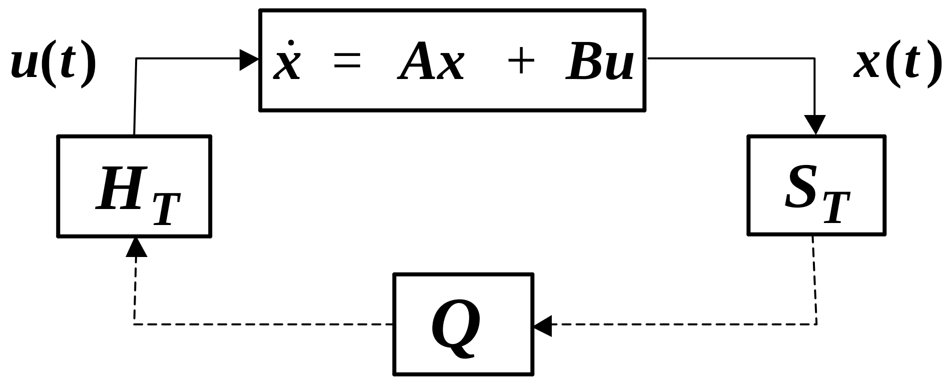
Introduction to probabilistic approach

- Studied in robust control
- Low-complexity randomized algorithms
- Probabilistic information with “small” risk

Analysis/design through randomized algorithms

- Calafiore and Polyak (2001)
- Polyak and Tempo (2001)

2. Problem formulation

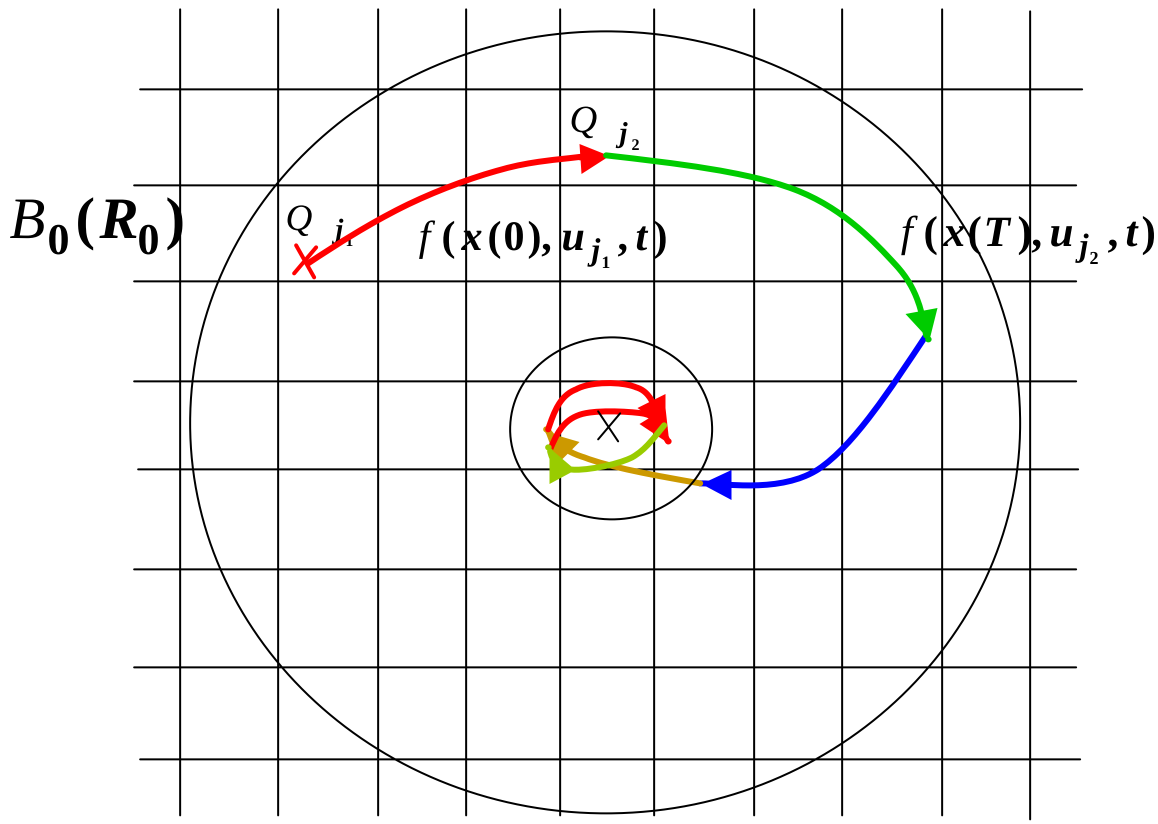


Partition $\{Q_j\}_{j \in S}$ of R^n

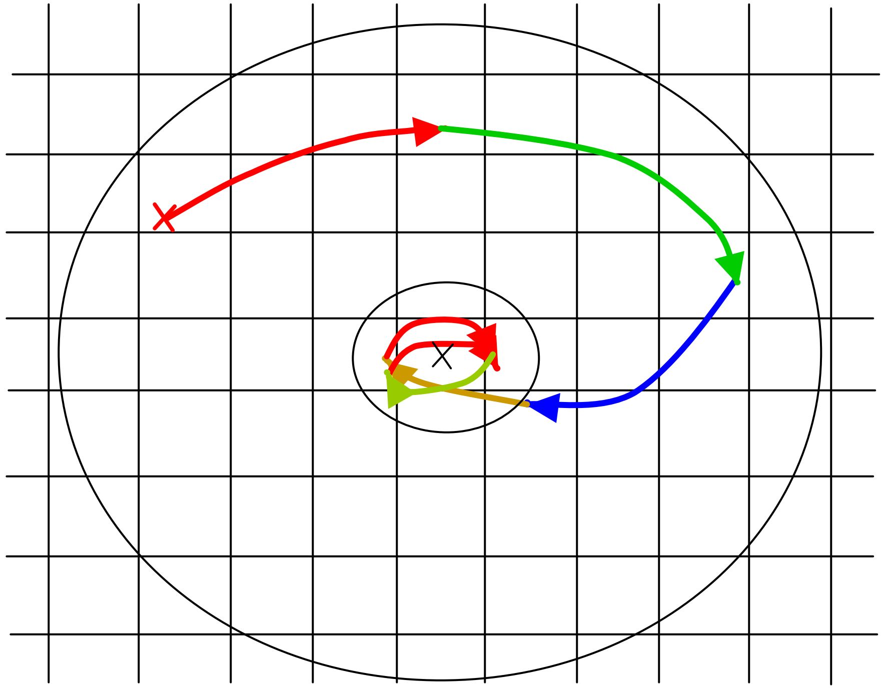
Output values $\{u_j\}_{j \in S} \subset R^n$

Quantizer $Q(x) = u_j$ if $x \in Q_j$

e.g. Uniform quantizer



Quadratic attractiveness



$B_0(r_0)$: Quadratically attractive wrt $V_P(x) = x'Px$

if for all $x(0) \in B_0(R_0)$,

$$\dot{V}_P(x(t),u(t)) \le -e\|x(t)\|^2 \quad \text{or} \quad x(t) \in \underline{E}_P^{r_0}, \text{ for } t \ge 0$$

where $\underline{E}_P^{r_0}$: level set of V_P contained in $B_0(r_0)$.

Problem

Given $B_{\mathbf{0}}(\boldsymbol{R}_{\mathbf{0}})$

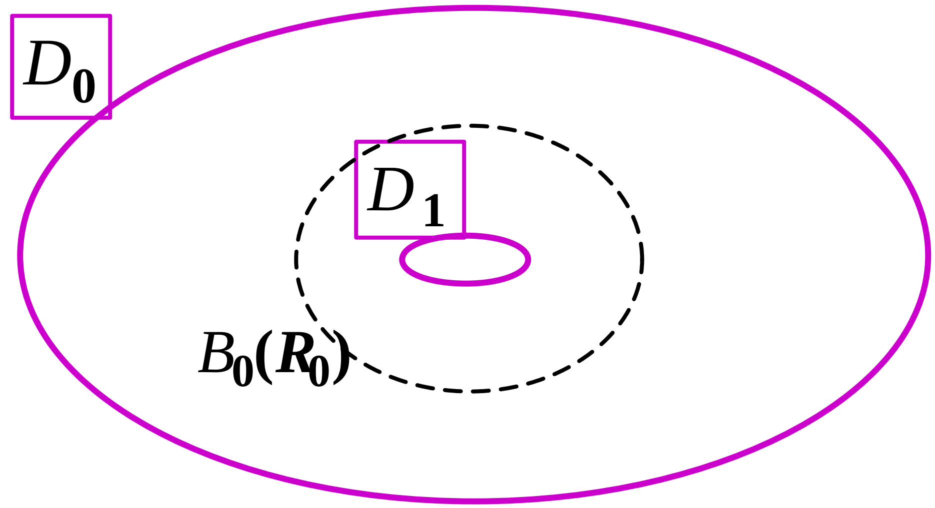
$e > \mathbf{0}$

$\boldsymbol{T} > \mathbf{0}$

$\boldsymbol{Q}:$ quantizer

Find $\boldsymbol{P} = \boldsymbol{P}' > \mathbf{0}$ and $\boldsymbol{r}_{\mathbf{0}} > \mathbf{0}$ for quadratic attractiveness

Sets D_0 and D_1



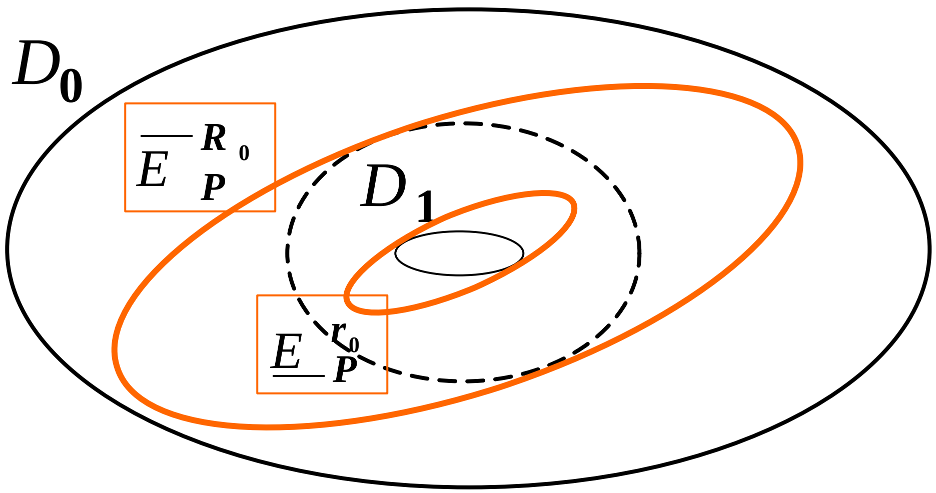
$S_N \subset S$: Indices of cells intersecting with D_0

A simple sufficient condition

1. $\forall\, x \in D_0,\ \forall\, t \in [0, T],$

$$\dot{V}_P(f(x, Q(x), t), Q(x)) \le -e \left\| f(x, Q(x), t) \right\|^2$$

or $f(x, Q(x), t) \in D_1.$



2. $D_1 \subset \underline{E}_P^{r_0} \subset \overline{E}_P^{R_0} \subset D_0$

3. Probabilistic approach

A simple algorithm

1. Set initial $\boldsymbol{P}^{[0]}$.
2. Generate $\boldsymbol{x}^{[k]} \in D_{\mathbf{0}}$ and $\boldsymbol{t}^{[k]} \in [\mathbf{0}, \boldsymbol{T}]$ randomly.
3. Let $f^{[k]} = f(\boldsymbol{x}^{[k]}, \boldsymbol{u}_{\boldsymbol{j}^{[k]}}, \boldsymbol{t}^{[k]})$. Check if

$$\dot{\boldsymbol{V}}_{\boldsymbol{P}^{[k]}}(f^{[k]}, \boldsymbol{u}_{\boldsymbol{j}^{[k]}}) \le -\varepsilon \left\| f^{[k]} \right\|^{2}$$

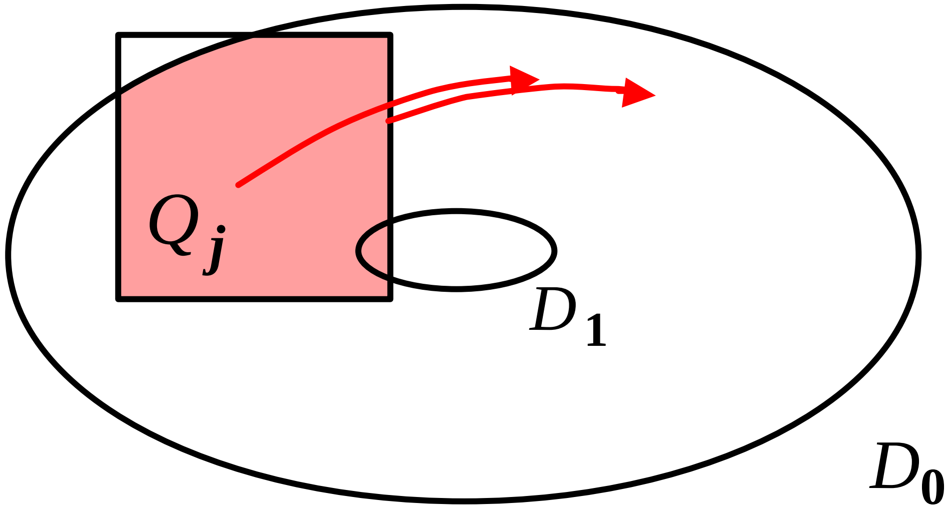
or $f^{[k]} \in D_{\mathbf{1}}$.

If not, then update $\boldsymbol{P}^{[k]}$.

A simple algorithm

- Gradient based
- Converges in a finite number of steps with probability one.
- Sytstematic way with less conservatism
- Randomization in x and t.

However, redundancy in the # of trajectories.



$$\forall \boldsymbol{j},\ \forall \boldsymbol{x} \in Q_{\boldsymbol{j}} \cap D_{\mathbf{0}},\ \forall \boldsymbol{t} \in [\mathbf{0},\boldsymbol{T}],$$

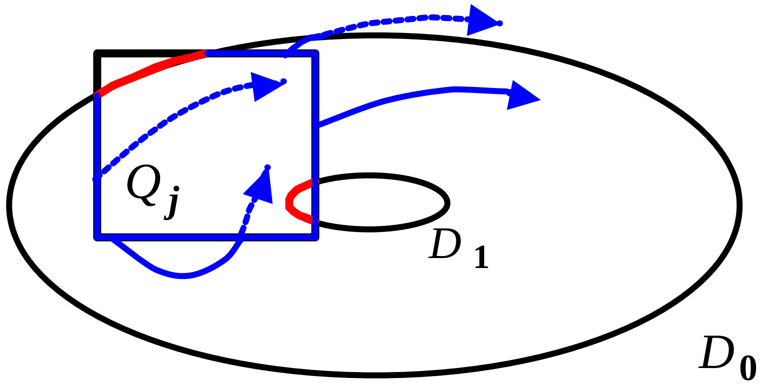
$$\dot{\boldsymbol{V}}_{\boldsymbol{P}}(f(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{j}}, \boldsymbol{t}), \boldsymbol{u}_{\boldsymbol{j}}) \le -\boldsymbol{\varepsilon} \left\| f(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{j}}, \boldsymbol{t}) \right\|^{\mathbf{2}} \text{ or } f(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{j}}, \boldsymbol{t}) \in D_{\mathbf{1}}.$$

How can we reduce the # of trajectories?

$$\left\{ \boldsymbol{x} : \dot{\boldsymbol{V}}_{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{j}}) > -\varepsilon \left\| \boldsymbol{x} \right\|^{\mathbf{2}} \right\}:$$

Each connected component is unbounded.

4. A modified sufficient condition



$F_{1j} = Q_j \cap (\text{boundaries of } D_0 \text{ and } D_1)$

$F_{2j} = (\text{boundary of } Q_j) \cap D_0$

1. $\forall \boldsymbol{j}, \forall \boldsymbol{x}$ on F_{1j}, $\dot{\boldsymbol{V}}_{\boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{u}_j) \le -\varepsilon \|\boldsymbol{x}\|^2$

2. $\forall \boldsymbol{j}, \forall \boldsymbol{x}$ on F_{2j}, and $\boldsymbol{t} \in [0, T]$,

   if $f(\boldsymbol{x}, \boldsymbol{u}_j, t), t \in (0, \boldsymbol{t}]$, does not enter $Q_j \setminus D_1$ or leave D_0,

   then $\dot{\boldsymbol{V}}_{\boldsymbol{P}}(f(\boldsymbol{x}, \boldsymbol{u}_j, \boldsymbol{t}), \boldsymbol{u}_j) \le -\varepsilon \|f(\boldsymbol{x}, \boldsymbol{u}_j, \boldsymbol{t})\|^2$ or $f(\boldsymbol{x}, \boldsymbol{u}_j, \boldsymbol{t}) \in D_1$.

3. $D_1 \subset \underline{E}_{\boldsymbol{P}}^{r_0} \subset \overline{E}_{\boldsymbol{P}}^{R_0} \subset D_0$

$\mathcal{P}$: The set of feasible matrices $\boldsymbol{P}$

Lemma:

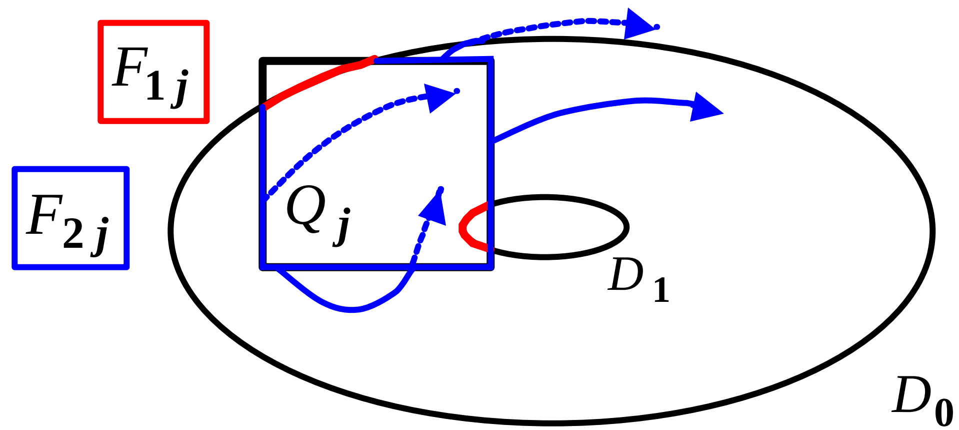
If $\boldsymbol{P} \in \mathcal{P}$ and $a > 1$, then $a\boldsymbol{P} + \Delta\boldsymbol{P} \in \mathcal{P}$ for all $\Delta\boldsymbol{P}$ with

$$\left\|\Delta \boldsymbol{P}\right\| \le (a - 1)\frac{\min_{\boldsymbol{x}\in\partial\boldsymbol{D}_1} e\left\|\boldsymbol{x}\right\|^{2}}{2\max_{\boldsymbol{x}\in\partial\boldsymbol{D}_0}\left\|\boldsymbol{x}\right\|(\max_{\boldsymbol{x}\in\partial\boldsymbol{D}_0}\left\|\boldsymbol{A}\boldsymbol{x}\right\| + \max_{\boldsymbol{j}\in S_N}\left\|\boldsymbol{B}\boldsymbol{u}_{\boldsymbol{j}}\right\|)}$$

Projection $[\cdot]^{+}$ of a matrix onto

the cone of nonnegative def. matrices

5. An efficient randomized algorithm



1. Set initial $\boldsymbol{P}^{[0,0]}$ and $\boldsymbol{\rho} > \mathbf{0}.$
2. Generate $\boldsymbol{j}^{[k]} \in S_N, \boldsymbol{x}^{[k]} \in F_{1\boldsymbol{j}^{[k]}} \cup F_{2\boldsymbol{j}^{[k]}}$ randomly.

   a. If $\boldsymbol{x}^{[k]} \in F_{1\boldsymbol{j}^{[k]}}$ and if $\dot{V}_{\boldsymbol{P}^{[k,0]}}(\boldsymbol{x}^{[k]}, \boldsymbol{u}_{\boldsymbol{j}^{[k]}}) > -\varepsilon \left\| \boldsymbol{x}^{[k]} \right\|^2$

   then update $\boldsymbol{P}^{[k,0]}$ by

$$\boldsymbol{P}^{[k+1,0]} = \left[\boldsymbol{P}^{[k,0]} - \mu^{[k,0]} \nabla_{\boldsymbol{P}} \dot{V}_{\boldsymbol{P}^{[k,0]}}(\boldsymbol{x}^{[k]}, \boldsymbol{u}_{\boldsymbol{j}^{[k]}}) \right]^{+}.$$

Here $\mu^{[k,0]} = \dfrac{\dot{V}_{P^{[k,0]}} + \varepsilon \left\| x^{[k]} \right\|^2 + \rho \left\| \nabla_P \dot{V}_{P^{[k,0]}} \right\|}{\left\| \nabla_P \dot{V}_{P^{[k,0]}} \right\|}$.

b. If $x^{[k]} \in F_{2 j^{[k]}}$, then generate $0 = \tau^{[k,0]} < \cdots < \tau^{[k,l]} = T$.

If $\tau^{[k,i]} \neq 0$ and $f(x^{[k]}, u_{j^{[k]}}, \tau^{[k,i]}) \in \mathrm{cl}(Q_{j^{[k]}} \cup D_0^c \setminus D_1)$,

then go to next step. Otherwise, update $P^{[k,i]}$.

c. Set $P^{[k+1,0]} = P^{[k,l]}$.

3. Find $r_0 > 0$ such that $D_1 \subset \underline{E}^{r_0}_{P^{[k,0]}} \subset \overline{E}^{R_0}_{P^{[k,0]}} \subset D_0$.

Theorem:

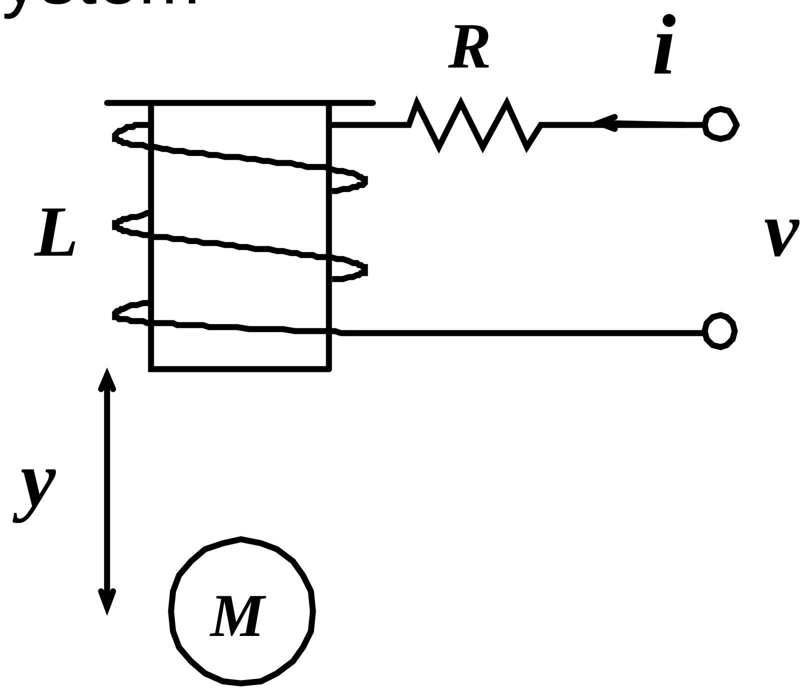
Suppose *P* is nonempty. Then, the algorithm converges in finite steps with probability one.

Comments

- Probabilistic result: Random generation of x and t
- Rate of convergence
- # of steps: Difficult to estimate
- Parameter r: Determines step size; Arbitrary.

6 . Examples

Magnetic ball system



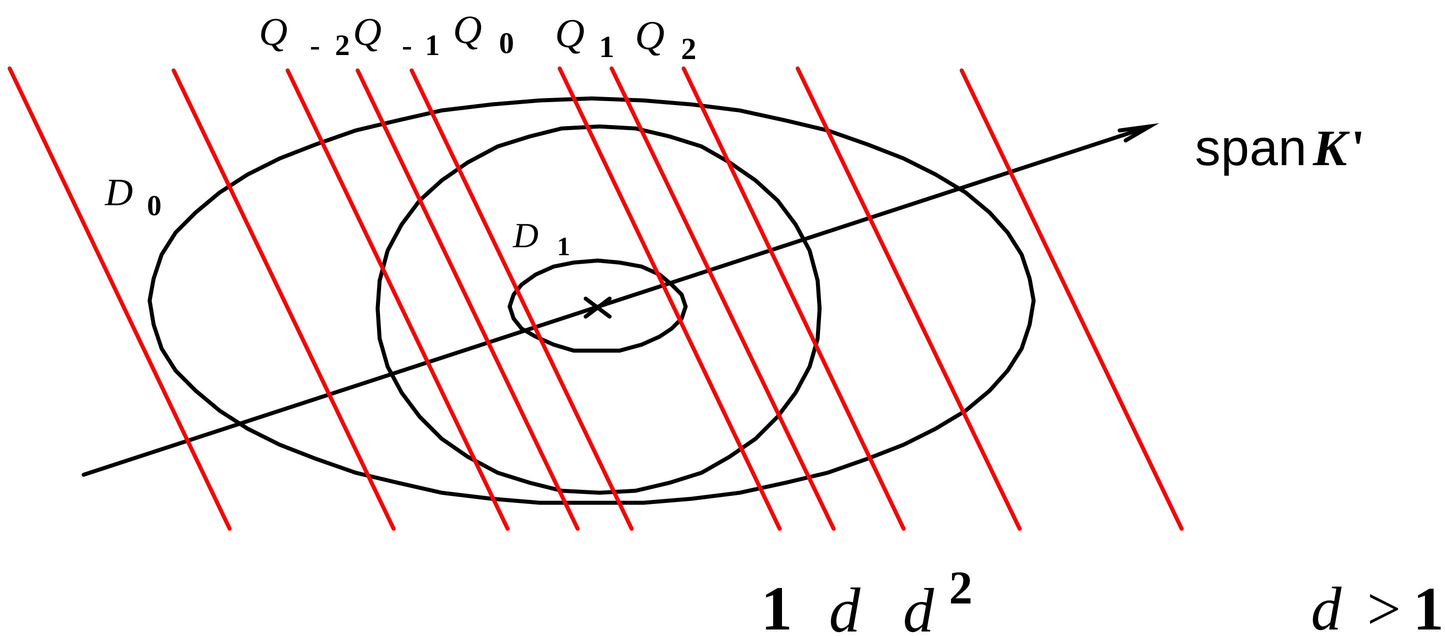
State : $\boldsymbol{x} = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix}$

Design of Q and T [Ishii and Francis, 2002]

K : LQR optimal with $P_0 > 0$

Logarithmic Q and T :

Quadratically stable wrt $V_{P_0}(x) = x' P_0 x$

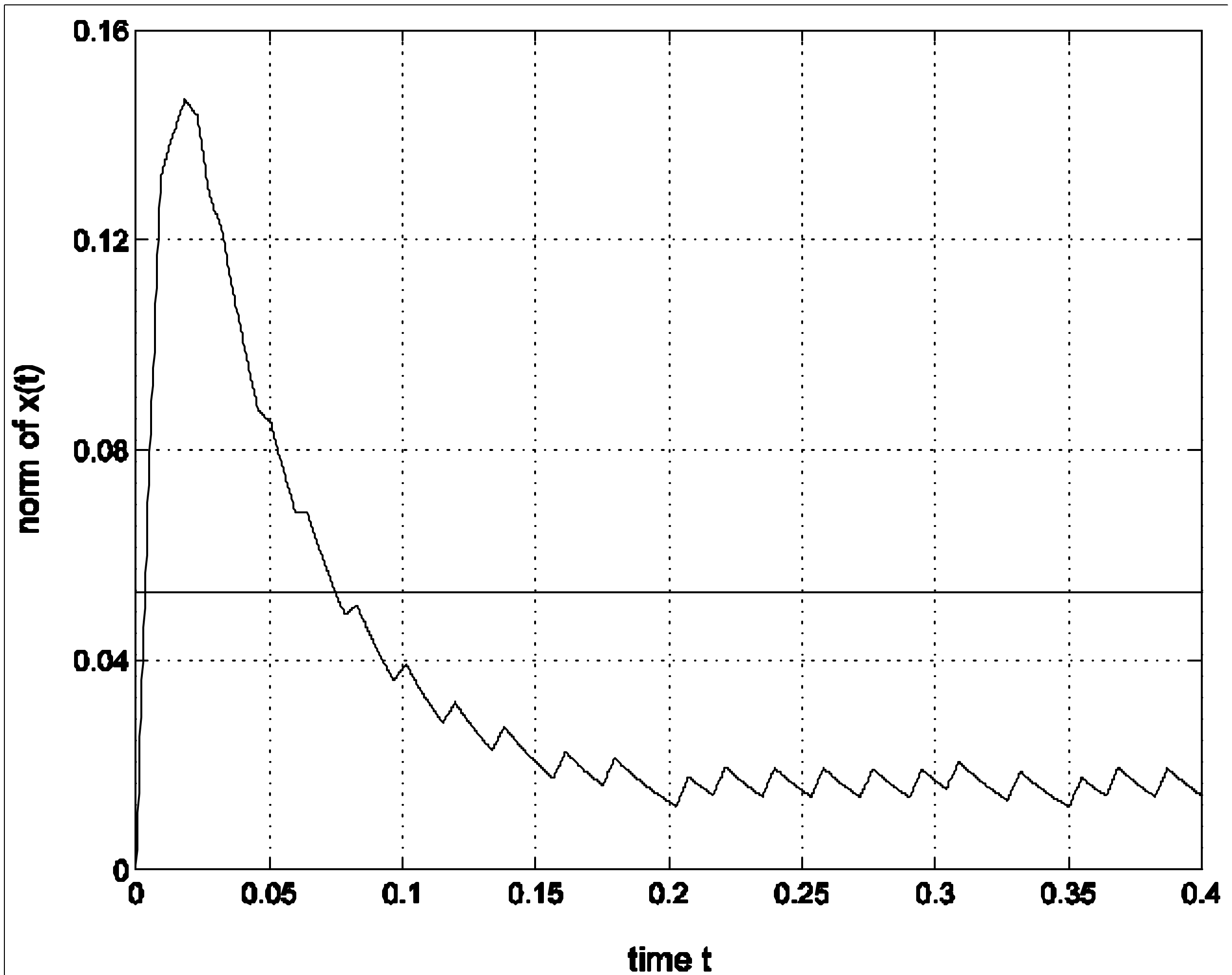


- In design, $\boldsymbol{R_0} = \mathbf{10}$ and $\boldsymbol{r_0} = \mathbf{32}$
- In simulation, $\boldsymbol{r_0} = \mathbf{0.02}$ approx.
- $\boldsymbol{T} = \boldsymbol{T_0} = \mathbf{0.003}$

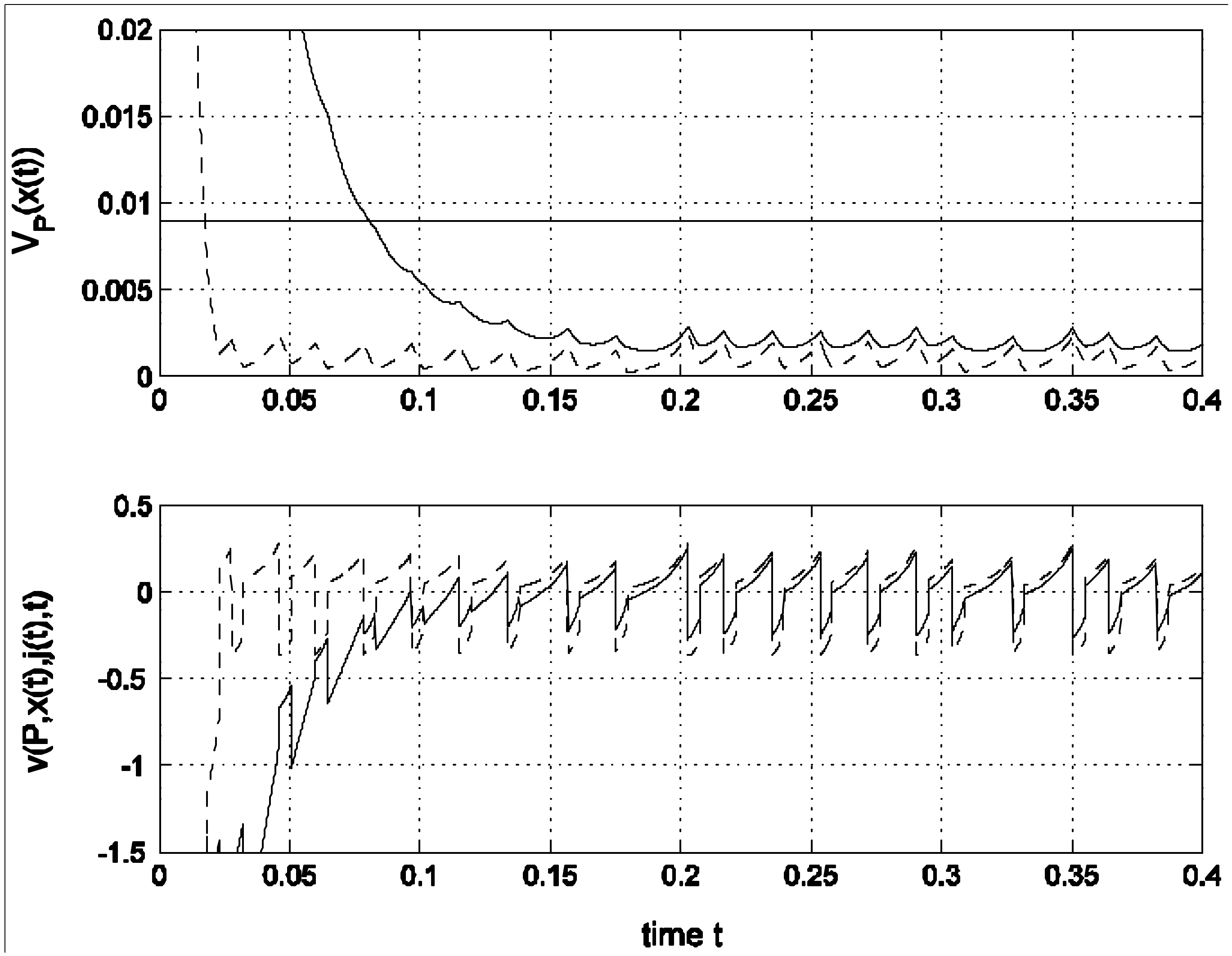
Results using randomized algorithm

- For $\boldsymbol{T} = \mathbf{1.5} \times \boldsymbol{T_0}$, obtained $\boldsymbol{r_0} = \mathbf{0.053}$.
- Random samples : 190,000 in $\boldsymbol{x}$

  4 in $\boldsymbol{t}$ for each $\boldsymbol{x} \in F_{\mathbf{2}\boldsymbol{j}}$
- # of updates : 28



0.16
0.12
0.08
0.04
0
norm of x(t)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
time t



0.02
0.015
0.01
0.005
0
V_P(x(t))
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.5
0
-0.5
-1
-1.5
v(P,x(t),j(t),t)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
time t

7. Conclusion

- A randomized algorithm for analysis of quantized sampled-data systems
- Converges with probability one
- Based on a new sufficient condition
- Computationally efficient